Q-ary collision resolution algorithms in random-access systems with free or blocked channel access

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Résumé: Les protocoles en arbre du type Capetanakis-Tsybakov-Mikhailov-Vvedenskaya permettent de gérer de manière distribuée l'accès à un canal à accès multiple. On étudie dans cet article l'effet de branchements "Q-aires" où Q est un paramètre utilisé pour séparer les groupes d'utilisateurs entrés en conflit. Différents algorithmes sont analysés et l'on montre que l'emploi d'un branchement ternaire au lieu du branchement binaire habituel conduit à une optimisation aussi bien en terme de délai que de capacité de transmission. Ceci permet d'atteindre, avec un protocole à accès libre, une capacité de transmission sur le canal égale à 0.4016.

Abstract: Tree protocols of the Capetanakis-Tsybakov-Mikhailov-Vvedenskaya type are encountered in the distributed management of a single communication medium. We study here the effect of "Q-ary" branching where Q is a parameter used for splitting contending users. Several algorithms are analysed and we show that using ternary splitting (instead of a customary binary splitting) results in an absolute optimization both in terms of throughput and packet delay; the maximal throughput attainable in this way with a free access protocol is equal to 0.4016 packets per slot.

Q-ary Collision Resolution Algorithms in Random-Access Systems with Free or Blocked Channel-Access

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Abstract

Q-ary tree or stack algorithms - $Q \geqslant 2$ is a parameter used for splitting contending users - of the Capetanakis-Tsybakov-Mikhailov-Vvedenskaya type, as encountered in the distributed management of a single communications medium with an infinite number of users, are analyzed. Both, free and blocked channel-access protocols are considered and combined with Q-ary collision resolution algorithms using either binary or ternary feedback. For these algorithms functional equations for the generating functions of all moments of the collision resolution interval length are obtained. The maximum stable throughput as a function of Q is given and it is shown that the favourite algorithm (in terms of ease of implementation) uses ternary splitting and binary feedback while allowing free channel-access, thereby yielding a maximum stable throughput of .4016 packets per slot when the new packet process is Poisson.

I. Introduction

The problem considered (see Fig. 1.1) is the random-accessing by an infinite population of transmitters of a common time-slotted collision-type channel with either

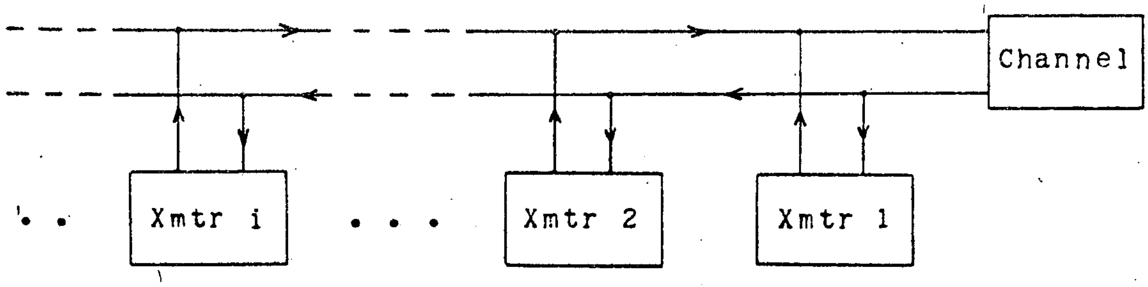

Fig. 1.1 Diagram of the system

noiseless binary feedback ("collision" or "no collision"), or noiseless ternary feedback ("idle", "success" or "collision").

All transmitters are assumed to be identical and to transmit independent information. The transmitters are constrained to transmit data only in the form of "packets" whose length is one time slot. Moreover, the transmitters are time-synchronized and obliged to start their transmissions exactly at the beginning of a slot. When more than one transmitter sends a packet in the same slot, a "collision" among the packets occurs; it is assumed that all packets involved in that collision are completely destroyed and lost. Immediately at the end of each slot, all transmitters are informed simultaneously whether that slot contained a collision or not, or, for the ternary feedback, whether that slot was empty ("idle"), contained a single packet ("success") or suffered a collision.

The real issue in any random-access system (RAS) of the above type is the resolution of conflicts for the use of the common medium. Beginning in 1970 with publication of the ALOHA-algorithm by Abramson [ABR70], a huge literature has sprouted, devoted to the analysis of the performance of random-access schemes. Until quite recently, most of this work consisted of variations on the original ALOHA system although, unfortunately, this system is inherently unstable in the absence of external control when the number of stations is large - for a more detailed survey, see [KLE76]. The next major innovation came in 1977 with the introduction by Capetanakis [CAP77] of the concept of a collision resolution algorithm (CRA). Tsybakov and Mikhailov [TSYMIK78] later, but independently, proposed the use of CRA's and advanced their analysis - for a rather complete survey, see [MAS81]. The use of CRA's was shown by Capetanakis to result in stable RAS's when the packet arrival rate is not too high; the key parameter of the system is then its maximum stable throughput.

A CRA can be defined as a distributed algorithm that organizes the retransmissions of packets in such a way that, after every initial collision of first-time-sent packets, (provided in some cases that the packet arrival rate is not too high) each of the colliding packets and each later first-time-sent packet (if any) is eventually sent successfully and, moreover, all transmitters simultaneously become aware of this fact. The time span from the slot where the initial collision occurred up to and including the slot from which all transmitters recognized that all colliding and later first-time-sent packets had been successfully received is called a <u>collision resolution interval</u> (CRI). We say that a CRA is Q-ary (where Q is normally an integer greater or equal to 2) if a transmitter after being in a collision must choose randomly among Q possible further courses of action.

In order to form a RAS, a CRA must be used together with a <u>channel-access protocol</u> (CAP). The CAP is the distributed algorithm that determines for each transmitter when a newly arrived packet at that transmitter is sent for the first time. The simplest CAP, both conceptually and practically, is the free access protocol (FAP) in which a transmitter sends a new packet in the first slot following its arrival. All other CAP's will be called blocked access protocols (BAP's). The obvious BAP is that in which a transmitter sends a new packet in the first slot following the CRI in progress upon its arrival. All other BAP's will be called non-obvious.

An individual transmitter can be in one of three main states (see Fig. 1.2). Transmitters who have no packet to

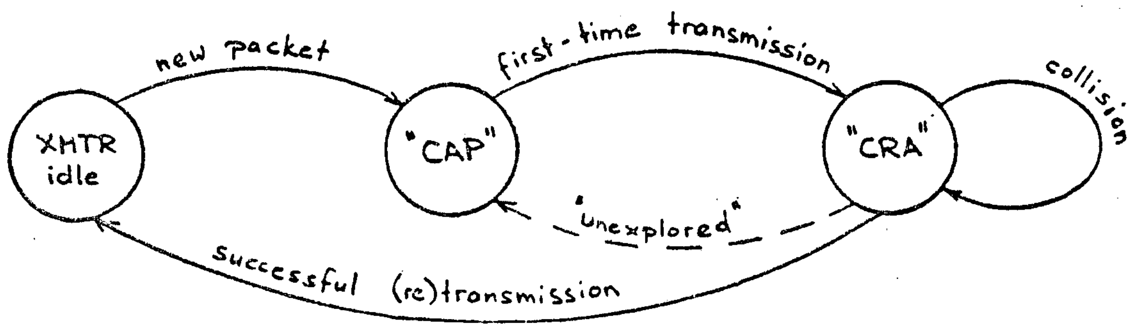

Fig. 1.2 Transmitter state diagram. The dashed arrow leading back from the "CRA" to the "CAP" state refers to certain non-obvious BAP's (e.g. [GAL78] or [TSY-MIK80]) which are not treated in this paper.

transmit (or retransmit) are in an idle state which in the case of BAP's does not necessarily relieve them of observing the CAP or the CRA currently in progress. When a transmitter obtains a new packet, it moves to the "CAP" state, i.e., it follows the CAP (which might involve observing the CRA in progress) until that packet is sent for the first time. If a collision occurs, then this transmitter follows the CRA ("CRA" state) until that packet is successfully retransmitted.

This paper will be concerned with the analysis of RAS's consisting of a Q-ary CRA and a CAP of either the free access or obvious blocked access type. For two reasons, non-obvious BAP's will not be considered although, as will be discussed later, these BAP's offer the highest stable throughputs presently known. The first reason is the practical one that the complexity of implementing the protocol and its susceptibility to disturbance by channel error conditions militate against the use of such BAP's in real random-access situations. The second reason is that the emphasis of this paper is on Q-ary CRA's with $Q > 2$, and such nonbinary CRA's offer no advantage over binary CRA's when used with any of the non-obvious BAP's that have previously been introduced.

All of the CRA's considered previously have been binary. This exclusive focus on binary CRA's is perhaps due to the fact that Capetanakis [CAP77] proved that a binary CRA maximizes the maximum stable throughput when the CAP is always chosen optimally (e.g. after the termination of the preceding CRI) for RAS's of what he called the "dynamic tree algorithm" type. This seems to suggest that binary CRA's are generally optimum. In fact, however, we shall show that Q=3 is optimum for CRA's used with either the FAP or the obvious BAP †). We shall show further that the most attractive RAS from a practical viewpoint is a ternary CRA with free access requiring only binary feedback whose maximum stable throughput is .4016 packets per slot when the packet arrival process is Poisson. In order to demonstrate this superiority of ternary CRA's, we develop analytical techniques of considerable generality that should be of some interest in themselves (see for instance [FLA-SOT82] for other applications).

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†) There is one exception: RAS's which use the obvious BAP and a CRA with ternary feedback turn out to perform slightly better when Q=2 is chosen instead of Q=3, see Section III.

The paper is organized as follows. In Section II, we introduce the four specific RAS's that will subsequently be analyzed. In Section III, we treat those RAS's which use the obvious BAP. Sections IV and V deal with the analysis of RAS's which make use of the FAP. Finally, in Section VI, conclusions are drawn from the preceding analyses.

II. Specification of the Algorithms

In this section we specify the CRA's and the CAP's and define some terms needed for the understanding of the later computations.

The CAP states the rules for the first-time transmission of a newly arrived packet. Two different protocols are considered.

a) Free access protocol. New packets are transmitted immediately at the beginning of the next slot following their arrival.

b) Obvious blocked access protocol. New packets are transmitted in the first slot after all previous conflicts are resolved, i.e., new packets remain blocked at their respective transmitters until the current CRI (if any) terminates.

In the sequel we assume for all algorithms that a transmitter has at most one packet on hand, either one involved in a previous collision or a new one waiting for its first transmission.

The CRA determines the actions to be taken to resolve collisions occurring on the channel. Two CRA's are treated in this paper.

c) Basic Q-ary CRA. After a collision each transmitter involved flips a "Q-sided coin" with values 1,2,...,Q. (The "Q-ary coins" need not be fair; we assume, however, that all coins are biased in the same way whenever biasing is used.) This splits the set of contending transmitters into Q subsets, according to the value each one flipped. Transmitters already assigned to subsets because of previous collisions increase their subset indices by Q when the collision occurs. Transmitters in subset 1, together with transmitters having new packets who are permitted by the CAP to send, send in the very next slot. When a success occurs, all transmitters assigned to subsets decrease their subset indices by one. Again those in subset 1, together with transmitters having new packets who are permitted by the CAP to send, send in the very next slot. Thus, if after an initial collision of two or more packets, each of the Q resulting subsets contains at most one transmitter, then the collision will be resolved after exactly Q slots following the initial collision, i.e., the CRI will be Q+1 slots long. The CRI-length when zero or one packets "collide" is, by definition, one slot. The basic CRA distinguishes only "collision" and "no collision", therefore binary feedback suffices.

d) Modified Q-ary CRA. The basic collision resolution mechanism is the same as described under c), except that the ternary feedback information, which distinguishes slots containing exactly one transmitter from idle slots, is exploited. If after a collision the next Q-1 slots turn out to be empty (implying no new arrivals during Q-1 slots if the free access protocol a) is used), then the next slot (corresponding to subset Q) must contain a collison if the basic CRA is used. This otherwise-wasted slot can be skipped by having all transmitters immediately act as if it had occurred. It should be noted, however, that in the presence of channel errors, the modified CRA can suffer from deadlock when used with with any of the BAP's [MAS81].

The combinations of a) and b) with c) and d) result in four different RAS's.

e) Using b) and c) results in the basic blocked access RAS's (basic CRA, obvious BAP). This is, from a computational point of view, the simplest case. The original source in the binary case is [CAP77] and [CAP79], in the Q-ary case [MAT82]. For a flowchart and an example of a basic blocked access RAS, see Figs. 2.1 and 2.3, respectively.

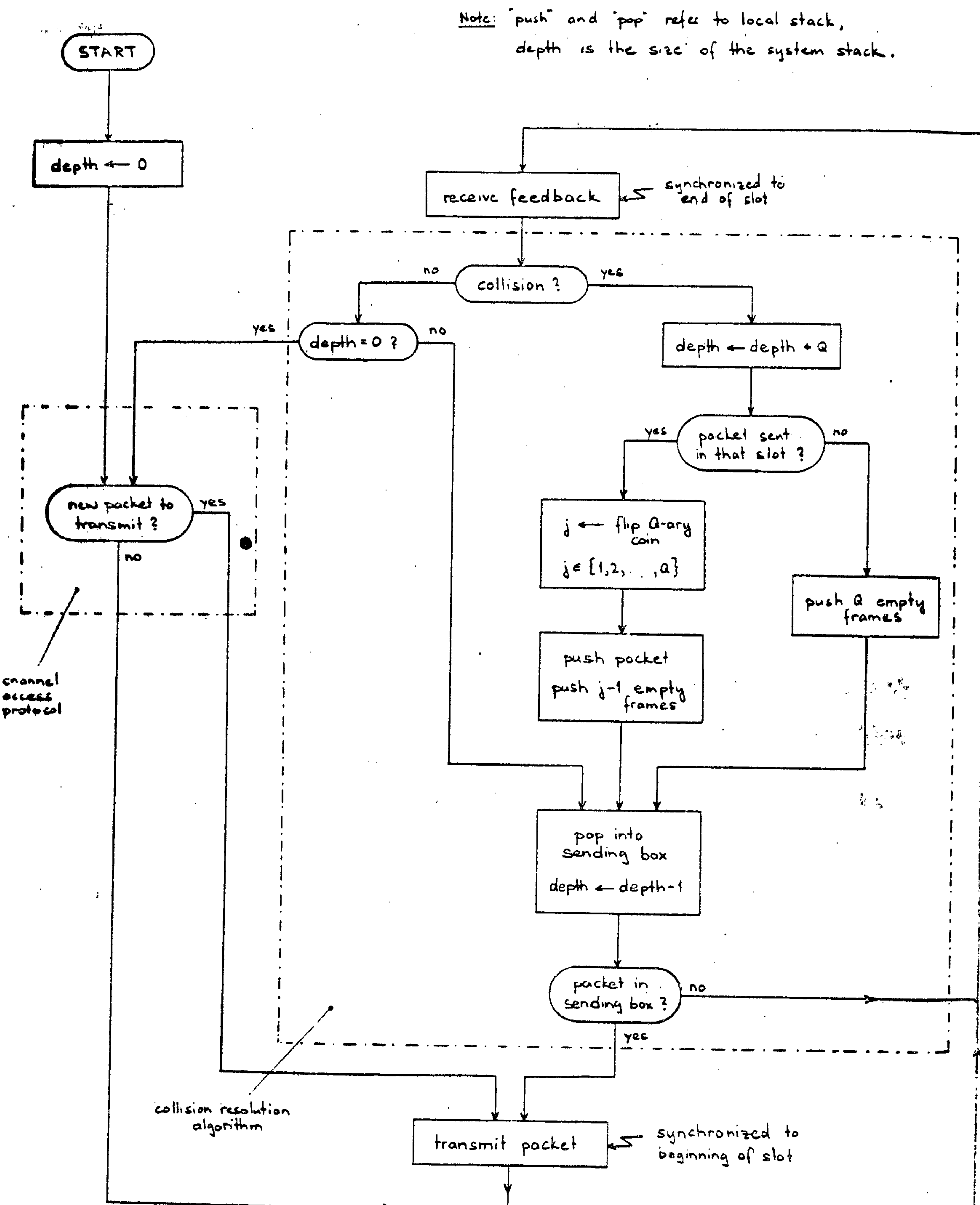


Fig. 2.1 Flowchart for the transmitters in a basic blocked access RAS.

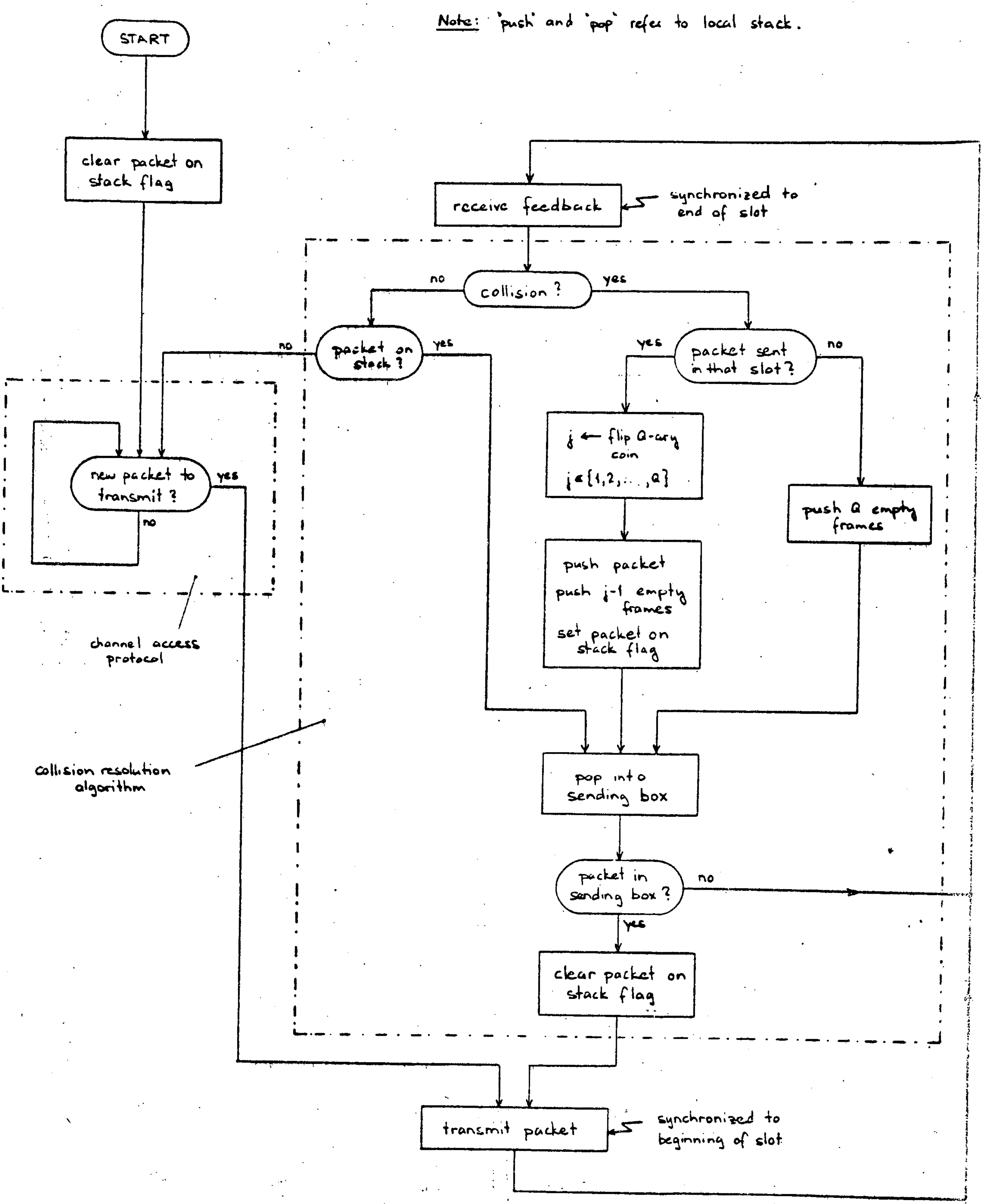
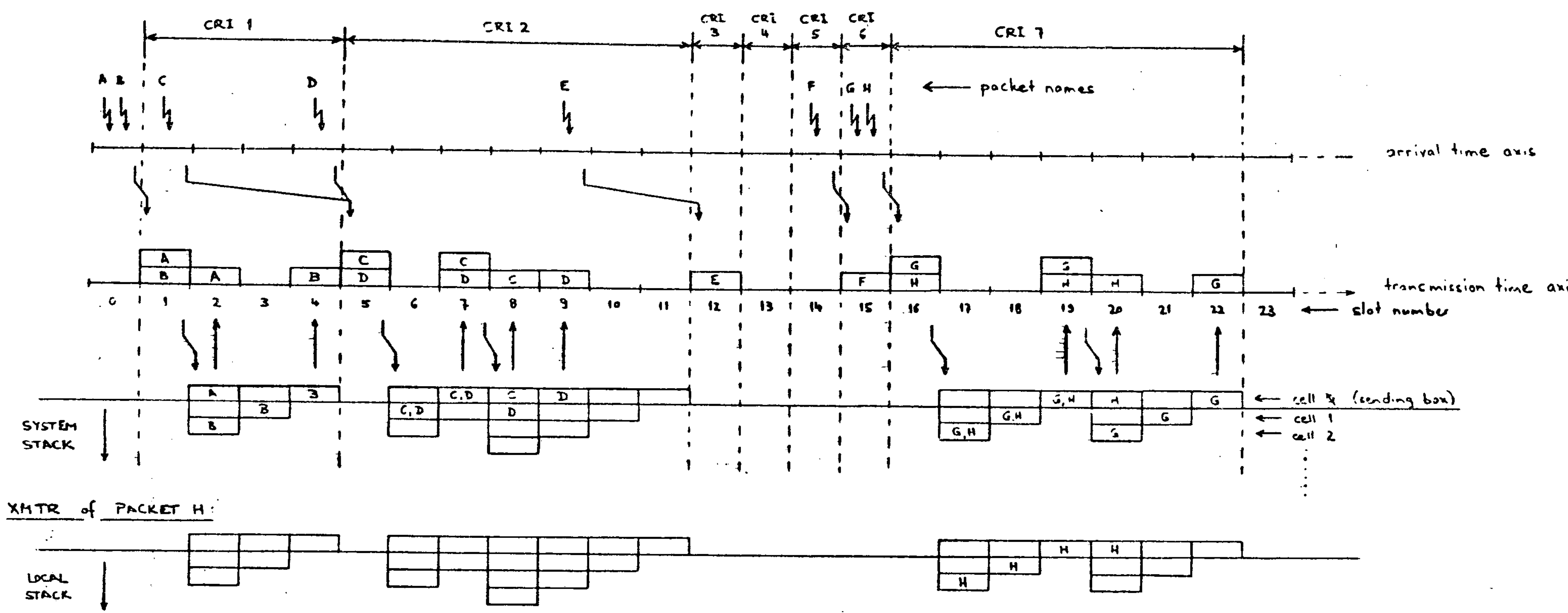
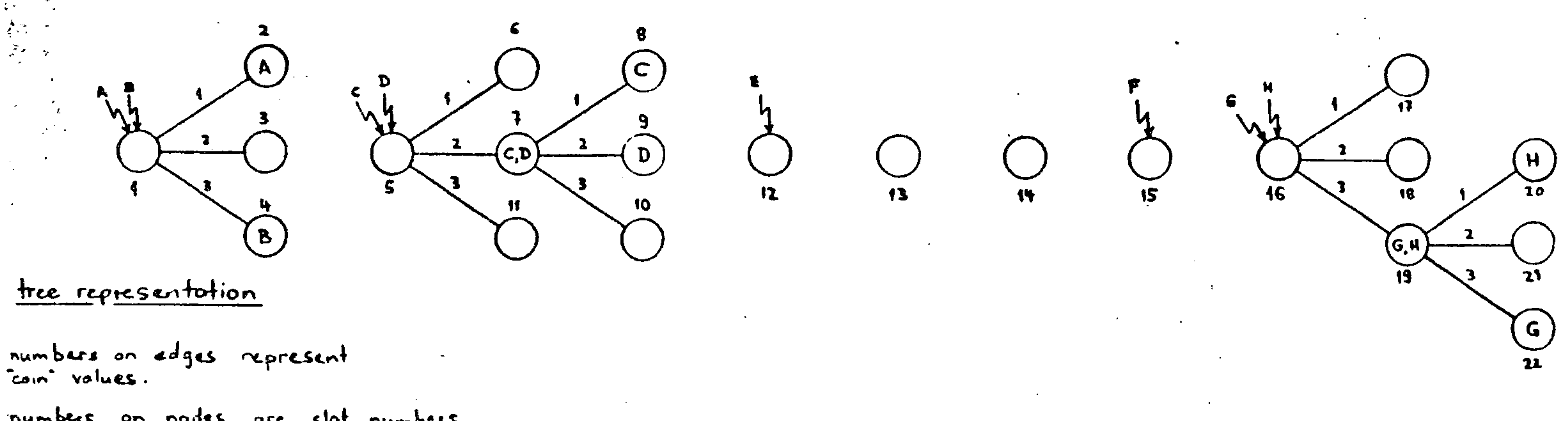


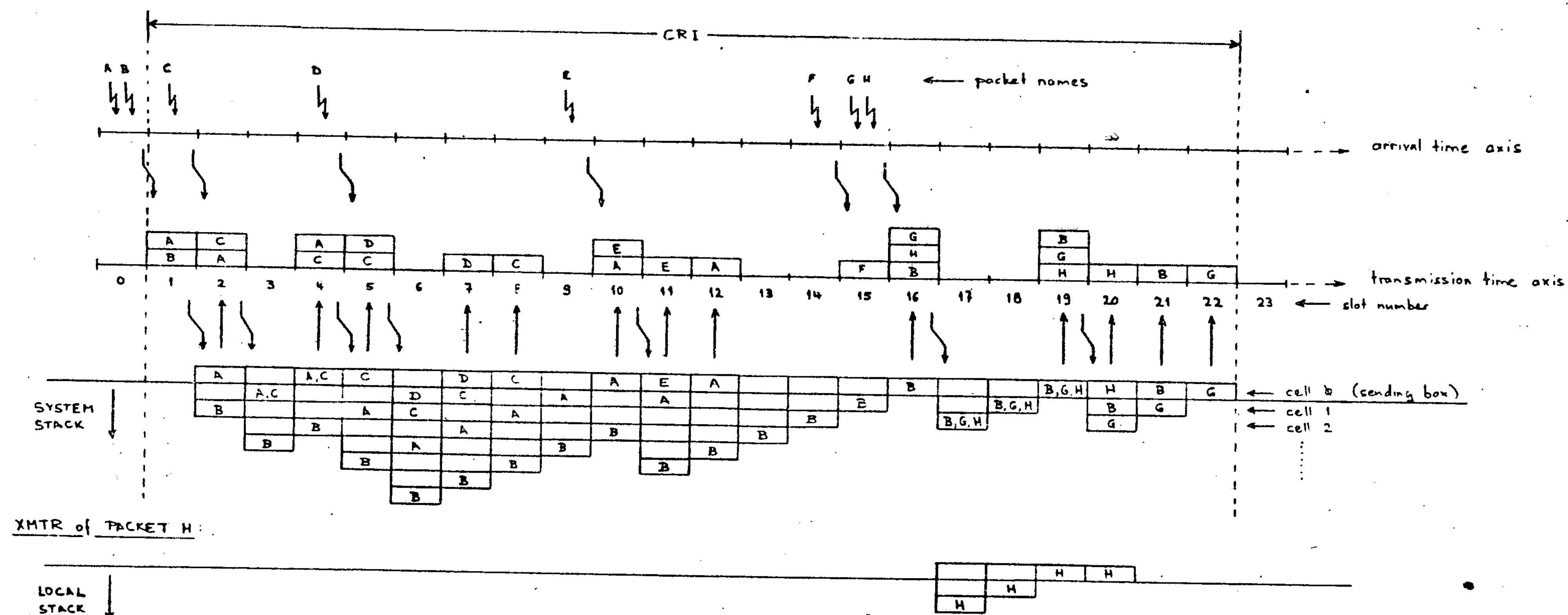
Fig. 2.2 Flowchart for the transmitters in a basic free access RAS.

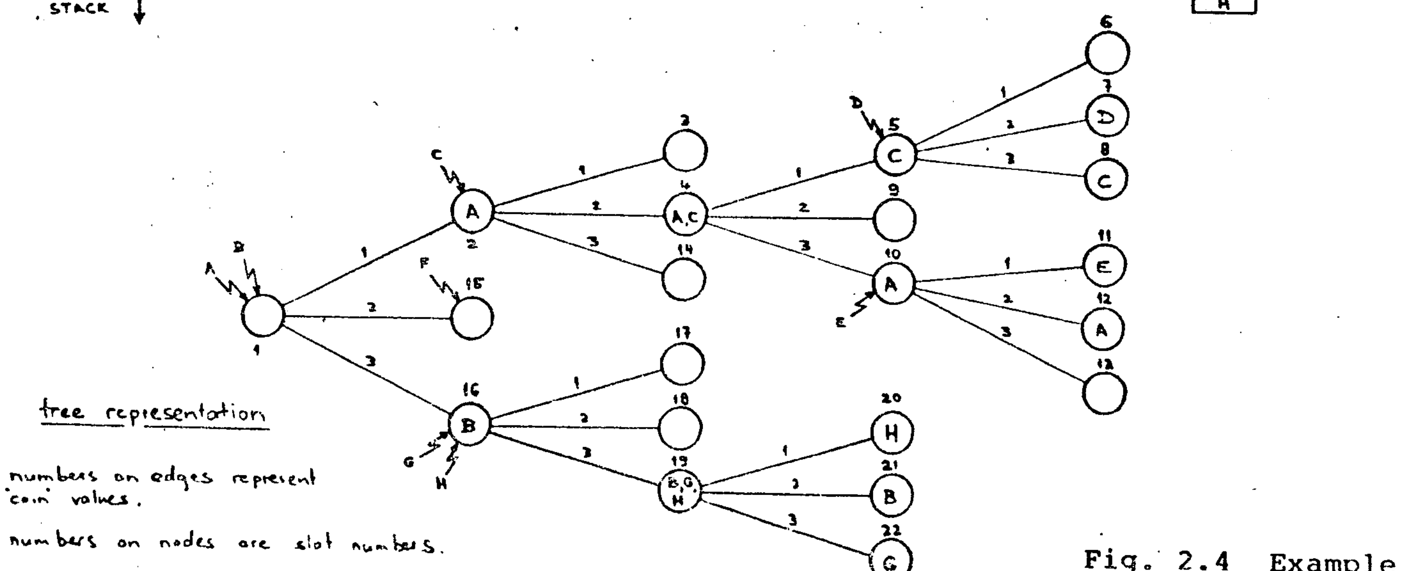




packet	coin tossing sequence of associated transmitter
A	1, ... -
B	3, ...
C	2, 1, ...
D	2, 2, ...
E	...
F	...
G	3, 3, ...
H	3, 1, ...

Fig. 2.3 Example of the transmission process in a basic blocked access RAS with Q = 3.





pocket	coin tossing sequence of associated transmitter
A	1, 2, 3, 2, ...
B	3, 3, 2, ...
C	2, 1, 3, ...
D	2, ...
E	1, ...
F	......
G	3, 3, ...
H	3, 1, ...

Fig. 2.4 Example of the transmission process in a basic free access RAS with Q = 3.

f) Combining b) and d) yields the <u>modified blocked access RAS's</u> (modified CRA, obvious BAP). The original sources when Q=2 are independently [TSY-MIK78] and Massey in [CAP79] and [MAS81], some remarks concerning the Q-ary case when Q>2 are in [MAT82]. Refering to Fig. 2.3, a modified blocked access RAS would save slot 19.

g) The combination a) and c) produces the <u>basic free access RAS's</u> (basic CRA, FAP). Computationally, these RAS's cannot be treated recursively anymore, as was possible with the blocked access RAS's. This is due to the free access of newly arrived packets which can cause the number of contending transmitters to increase during a CRI. In principle, the original source of the basic free access RAS's with Q=2 is [TSY-VVE80], computations in this particular case were first published in [FAY-HOF83] and [FAY-FLA-HOF82]. Later on, and independent of our investigations, basic free access RAS's with Q=2 and Q=3 were also treated in [VVE-TSY83]. For a flowchart and an example, see Figs. 2.2 and 2.4.

h) Finally, a) together with d) gives the <u>modified free access RAS's</u> (modified CRA, FAP). In terms of mathematical treatability, this is by far the most complicated class of RAS's. The original source when Q=2 is [TSY-VVE80], a mathematically different approach can be found in [FAY-HOF83]. Refering to Fig. 2.4, a modified free access RAS would save slot 19.

The flowcharts of Figs. 2.1 and 2.2 and the examples in Figs. 2.3 and 2.4 use a stack model to describe the respective RAS's.

i) Each transmitter has his own <u>local stack</u> consisiting of cells 0,1,2,... (In practice, each transmitter keeps only a counter that gives the size of his stack). Each cell can either store a packet or an empty frame (space holder). Cell 0 is at the top of the stack and is also called sending box. All other cells can only be accessed via push and pop instructions. Push moves the contents down by one cell, thereby transfering cell 0 to cell 1, pop is used to move the contents up, thereby transfering cell 1 to the sending sending box.

j) From a global point of view there is a conceptual quantity called the <u>system stack</u> which consists of all local stacks superimposed. The variable "depth" in Fig. 2.1 refers to the depth of the system stack which is the number of cells used (whether they be empty or contain a packet), excluding cell 0.

Note that, from an implementation point of view, the main difference between blocked access RAS's and free access RAS's is that for the former each transmitter has to maintain the global variable "depth" - i.e., monitor the channel without interruption - while for the latter transmitters only need local quantities - i.e., monitoring the channel by a transmitter is required only when that transmitter has a packet not yet sent successfully - which is an enormous practical advantage.

III. Analysis of RAS's with (obvious) blocked access

In this section we first derive the probability generating function (pgf) of Y_N - the CRI-length given N packets initially collide - directly from the specification of the blocked access RAS's. This pgf can then be used in a standard fashion to obtain equations (which are recursive in nature) for the first, second and, in principle, all higher moments of Y_N. The first moment $E[Y_N]$ will be denoted by L_N, the second moment $E[Y_N^2]$ by S_N. Next, we set up L(z) (and S(z)), exponential generating functions (egf's) for L_N (and S_N, respectively), in order to get a description of the moments of the CRI-length by way of functional equations. To obtain more convenient expressions (in terms of finding non-recursive solutions for the moments of Y_N) we subsequently use transformed (or Poisson) generating functions (tgf's), denoted by $L^*(z)$ (and $S^*(z)$, respectively). We then propose two methods of solution for the tgf's. One of them yields a solution which will be amenable to an analysis of the asymptotic behaviour of L_N (and, in principle, of S_N) as $N \to \infty$ which will be essential for the determination of the maximum stable throughput of the blocked access RAS's.

Fig. 3.1 visualizes the CRA for the blocked access RAS's. N is the number of packets that initially collide and I_j is the (random) number of corresponding transmit-

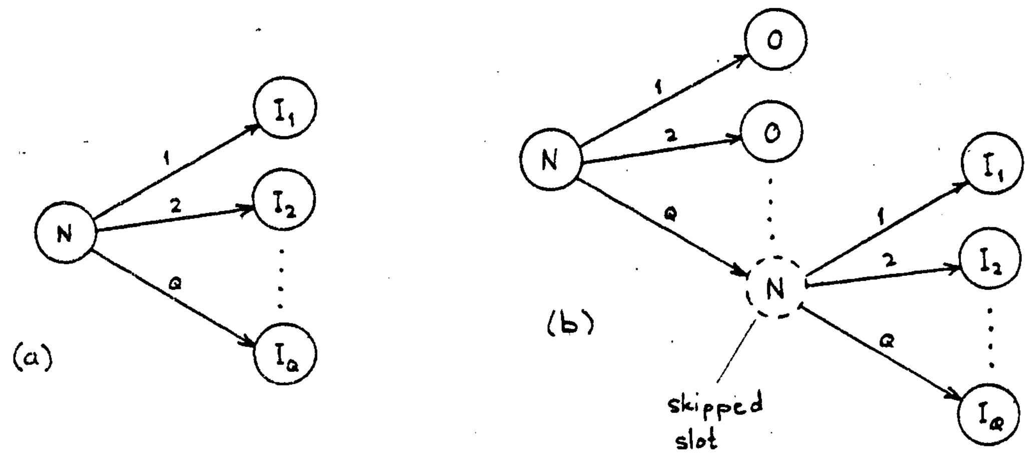


Fig. 3.1 Splitting N packets to Q subsets. (a) represents one stage of the "coin" flipping process in the normal case. (b) shows two stages of the "coin" flipping process in the modified case.

ters that flipped the value j on their "coin", $j \in \{1, 2, \ldots \ldots, Q\}$. From Fig. 3.1 we obtain directly the following recurrences, for the normal case

$$Y_N = 1 + \sum_{j=1}^{Q} Y_{I_j} \qquad , \qquad N \geqslant 2 \quad , \tag{3.1}$$

and for the modified case

$$Y_N = 1 + (Q-1) \cdot Y_0 + (Y_N - 1) \qquad , \qquad N \geqslant 2 \quad , \tag{3.2}$$

with the initial values $Y_0 = Y_1 \triangleq 1$. (3.3)

Define the pgf of Y_N as [FEL68]

$$G_N(s) \triangleq \sum_{k=0}^{\infty} \Pr\{Y_N = k\} \cdot s^k = E[s^{Y_N}] = E[s^Y | N] \quad . \tag{3.4}$$

By the standard technique of further conditioning the expectation on the RHS of (3.4) we get

$$G_N(s) = E[\ E[s^Y | I_1 \ldots I_Q , N] \ | N] \qquad , \tag{3.5}$$

which, upon substituting (3.1) and noting that the size of the j-th subtree in Fig. 3.1 depends only on I_j, yields the following relation between pgf's for the basic CRA

$$G_N(s) = s \cdot \sum_{i_1 \ldots i_Q}^{N} \binom{N}{i_1 \ldots i_Q} \cdot \prod_{j=1}^{Q} p_j^{i_j} \cdot G_{i_j}(s) \quad , \quad N \geq 2 \quad , \qquad (3.6)$$

where $\sum\limits_{i_1 \ldots i_Q}^{N} \triangleq$ sum over all possible combinations of $i_1 \ldots i_Q$ such that $\sum\limits_{j=1}^{Q} i_j = N$ and $i_j \geq 0$, (3.7)

$\binom{N}{i_1 \ldots i_Q} \triangleq$ multinomial coefficient, (3.8)

and $p_j \triangleq$ Pr{the j-th value of the "coin" is flipped} with $\sum\limits_{j=1}^{Q} p_j = 1$. (3.9).

The initial conditions (3.3) translate into

$$G_0(s) = G_1(s) = s \quad . \qquad (3.10)$$

Next, for the modified CRA, a term corresponding to the modified case (from equation (3.2)) has to be included in (3.6) which gives

$$G_N(s) = s \cdot \sum_{i_1 \ldots i_Q}^{N} \binom{N}{i_1 \ldots i_Q} \cdot \prod_{j=1}^{Q} p_j^{i_j} \cdot G_{i_j}(s) - \delta_M \cdot s^{Q-1} \cdot p_Q^N \cdot G_N(s) \cdot (s-1) \; , \quad N \geq 2 \; , \qquad (3.11)$$

with
$$\delta_M \triangleq \begin{cases} 0 & \text{if the basic CRA is used,} \\ 1 & \text{if the modified CRA is used.} \end{cases} \qquad (3.12)$$

To obtain an equation for L_N we simply take the first derivative of (3.11) with respect to s and evaluate it at s=1 which yields

$$L_N = 1 + \sum_{j=1}^{Q} \sum_{i_j=0}^{N} \binom{N}{i_j} \cdot p_j^{i_j} \cdot (1-p_j)^{N-i_j} \cdot L_{i_j} - \delta_M \cdot p_Q^N \quad , \quad N \geq 2 \; , \qquad (3.13)$$

with initial conditions from (3.10)

$$L_0 = L_1 = 1 \; . \tag{3.14}$$

Note that equations (3.13) and (3.14) could be used to compute L_N recursively (see for instance [MAS81]). In this paper we are interested, however, in the asymptotic behaviour of L_N as $N \to \infty$ and we will therefore require a direct expression for L_N. To this end we define a egf for L_N

$$L(z) \triangleq \sum_{N=0}^{\infty} L_N \cdot \frac{z^N}{N!} \; . \tag{3.15}$$

Substituting (3.13) in (3.15) and taking care of (3.14) yields

$$\begin{aligned} L(z) - 1 - z &= \sum_{N=2}^{\infty} (1-\delta_K p_a^N)\cdot\frac{z^N}{N!} + \\ &\quad + \sum_{j=1}^{Q}\sum_{N=2}^{\infty}\sum_{i_j=0}^{N}\binom{N}{i_j}\cdot p_j^{i_j}\cdot(1-p_j)^{N-i_j}\cdot L_{i_j}\cdot\frac{z^N}{N!} \\ &= \sum_{N=0}^{\infty}(1-\delta_K p_a^N)\cdot\frac{z^N}{N!} - (1-\delta_K) - z\cdot(1-\delta_K p_a) + \\ &\quad + \sum_{j=1}^{Q}\sum_{N=0}^{\infty}\sum_{i_j=0}^{N}\binom{N}{i_j}\cdot p_j^{i_j}\cdot(1-p_j)^{N-i_j}\cdot L_{i_j}\cdot\frac{z^N}{N!} + \\ &\quad - Q - \sum_{j=1}^{Q}(1-p_j)\cdot z - \sum_{j=1}^{Q} p_j\cdot z \; . \end{aligned} \tag{3.16}$$

From this we easily obtain a functional equation for $L(z)$

$$L(z) = e^{z}\cdot\sum_{j=1}^{Q} e^{-p_j z}\cdot L(p_j\cdot z) + e^{z} - Q\cdot(1+z) - \delta_K\cdot(e^{p_a z} - 1 - p_a\cdot z). \tag{3.17}$$

It will be more convenient for our subsequent computations to use a tgf, therefore we introduce

$$L^{*}(z) \triangleq \sum_{k=0}^{\infty} L_k^{*} \cdot z^k \triangleq e^{-z} \cdot L(z) \quad . \tag{3.18}$$

Note that (3.15) and (3.18) imply the following relation between L_N and L_k^{*}

$$L_N = \sum_{k=0}^{N} \frac{N!}{(N-k)!} \cdot L_k^{*} \quad . \tag{3.19}$$

Multiplying both sides of (3.17) by e^{-z} and substituting (3.18) we have

$$L^{*}(z) - \sum_{j=1}^{Q} L^{*}(p_j \cdot z) = Q \cdot f^{*}(z) + h^{*}(z) \tag{3.20}$$

where $\quad f^{*}(z) = - e^{-z} \cdot [1+z - \delta_M \cdot Q^{-1} \cdot (1+p_a \cdot z)]$, (3.21)

and $\quad h^{*}(z) = 1 - \delta_M \cdot e^{-z \cdot (1-p_a)}$, (3.22)

with initial conditions

$$L^{*}(0) = 1 \quad \text{and} \quad L^{*(1)}(0) = 0 \;^{\dagger)} \quad . \tag{3.23}$$

To obtain a similar tgf for the S_N (and, in principle, for all higher moments of Y_N) we first observe that

$$G_N^{(2)}(1) = S_N - L_N \quad . \tag{3.24}$$

Defining

$$S(z) \triangleq \sum_{N=0}^{\infty} S_N \cdot \frac{z^N}{N!} \quad , \tag{3.25}$$

and

$$S^{*}(z) \triangleq \sum_{k=0}^{\infty} S_k^{*} \cdot z^k \triangleq e^{-z} \cdot S(z) \quad , \tag{3.26}$$

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†) We use the definition $\Psi^{(n)}(a) \triangleq \left. \frac{d^n \Psi(z)}{dz^n} \right|_{z=a}$ throughout the paper to denote the n-th derivative of a function $\Psi(z)$ evaluated at z=a.

we obtain in the same manner as we did for (3.20) a functional equation for the tgf of the S_N

$$S^*(z) - \sum_{j=1}^{Q} S^*(p_j \cdot z) = Q \cdot f^*(z) + h^*(z) + \rho_L^*(z) \quad , \qquad (3.27)$$

where $\rho_L^*(z)$ is essentially a function of the conditional first moments of the CRI-length (see Appendix B) and $f^*(z)$ and $h^*(z)$ are defined as in (3.21) and (3.22). Note that the initial conditions for (3.27) are

$$S^*(0) = 1 \quad \text{and} \quad S^{*(1)}(0) = 0 \quad . \qquad (3.28)$$

We will now consider two basic methods to solve functional equations like (3.20) (and (3.27)). In the first one, which we will subsequently call the <u>direct method</u>, coefficients of z^k are equated on both sides of (3.20) which yields

$$L_k^* = \frac{Q \cdot f_k^* + h_k^*}{1 - \sum_{j=1}^{a} p_j^k} \quad , \quad k \geq 2 \quad , \qquad (3.29)$$

with $L_0^* = 1$, $L_1^* = 0$, (3.30)

and f_k^* and h_k^* defined as the coefficients of the power series representations (cf. (3.18)) of $f^*(z)$ and $h^*(z)$, respectively. Thus, together with (3.19), the mean value of the CRI-length, given N transmitters initially collided, is

$$L_N = 1 + \sum_{k=2}^{N} \binom{N}{k} \cdot \frac{(-1)^k \cdot \{Q \cdot (k-1) - \delta_M \cdot [k \cdot p_a + (1-p_a)^k - 1]\}}{1 - \sum_{j=1}^{a} p_j^k} \quad ,$$

$$N \geq 2 \; , \qquad (3.31)$$

with initial values (3.14). Note that the RHS of (3.31) is a series of alternating terms which makes numerical computations very sensitive as N becomes large.

The second method uses an iterative scheme (cf. Appendix A) to solve (3.20) (and (3.27)) directly for $L^*(z)$ ($S^*(z)$). In the sequel we refer to this method as the <u>iteration method</u>. In order to make the paper more readable, we restrict ourselves to CRA's which use fair coins, i.e., $p_j = Q^{-1}$, all $j \in \{1,2,\ldots,Q\}$. To obtain a functional equation which satisfies the contraction condition (cf. Appendix A), we proceed to differentiate (3.20) for fair coins twice wrt z, thus getting

$$L^{*(2)}(z) - Q^{-1} \cdot L^{*(2)}(z \cdot Q^{-1}) = Q \cdot f^{*(2)}(z) + h^{*(2)}(z) \quad . \qquad (3.32)$$

Here, $\lambda=0$ and therefore the m-th iterate of z is (from (A.4))

$$\sigma_H^{[m]}(z) = z \cdot Q^{-m} \quad . \qquad (3.33)$$

Now, using (A.11) we can solve (3.32) for $L^{*(2)}(z)$ which yields

$$L^{*(2)}(z) = \sum_{m=0}^{\infty} Q^{-m} \cdot [Q \cdot f^{*(2)}(\sigma_H^{[m]}(z)) + h^{*(2)}(\sigma_H^{[m]}(z))] \quad . \qquad (3.34)$$

To recover $L^*(z)$ we have to integrate (3.34) twice. Note that the initial conditions (3.23) determine the integration constants. To have a more convenient notation we define the operators $\mathfrak{R}$ and $\mathfrak{R}^{(1)}$ as

$$\mathfrak{R}^{(1)}(\Psi(.);z) \triangleq \sum_{m=0}^{\infty} [\Psi^{(1)}(\sigma_H^{[m]}(z)) - \Psi^{(1)}(\sigma_H^{[m]}(0))] \quad , \qquad (3.35)$$

$$\mathfrak{R}(\Psi(.);z) \triangleq \sum_{m=0}^{\infty} Q^{m} \cdot [\Psi(\sigma_H^{[m]}(z)) - \Psi(\sigma_H^{[m]}(0)) - Q^{-m} \cdot z \cdot \Psi^{(1)}(\sigma_H^{[m]}(0))] \, . \qquad (3.36)$$

Hence, integrating (3.34) yields

$$L^{*(1)}(z) = L^{*(1)}(0) + \mathcal{R}^{(1)}(Q\cdot f^{*}(\cdot) + h^{*}(\cdot);z) \quad , \qquad (3.37)$$

and finally, after integrating once more,

$$L^{*}(z) = L^{*}(0) + z\cdot L^{*(1)}(0) + \mathcal{R}(Q\cdot f^{*}(\cdot) + h^{*}(\cdot);z) \ . \quad (3.38)$$

The advantage of solution (3.38) over (3.29) is that in order to obtain an equation for L_N we can now use (3.18) to convert (3.38) directly into a relation for $L(z)$, i.e., (with (3.23) substituted)

$$L(z) = e^{z}\cdot L^{*}(z) = e^{z} + e^{z}\cdot\mathcal{R}(Q\cdot f^{*}(\cdot) + h^{*}(\cdot);z) \ . \quad (3.39)$$

By equating coefficients of $z^N/N!$ on both sides of this last equation we end up with

$$L_N = 1 + Q\cdot\sum_{m=0}^{\infty} Q^{m}\cdot[1-(1-Q^{-m})^{N} - N\cdot Q^{-m}\cdot(1-Q^{-m})^{N-1}] +$$
$$- S_N\cdot\sum_{m=0}^{\infty} Q^{m}\cdot\{[1-Q^{-m}\cdot(1-Q^{-1})]^{N} - (1-Q^{-m})^{N} - N\cdot Q^{-m-1}\cdot(1-Q^{-m})^{N-1}\},$$
$$N \geq 2 \ , \quad (3.40)$$

which is our desired non-recursive formula for L_N. The same procedure can be used to obtain an expression for S_N (see Appendix B) and, in principle, for all higher moments of Y_N.

Comments:

(1) From equation (3.31) one can immediately deduce that the first conditional moment of the CRI-length exists independent of the arrival process as long as $p_j < 1$

for all $j \in \{1,2,..Q\}$. This property can be shown to hold for all higher moments of Y_N since the denominator, which is the critical quantity in (3.31), does not change essentially for the higher moments (cf. (B.4) for the second moment).

(2) Assuming the new packet process to be Poisson with rate λ †), the blocked access RAS's will be stable (i.e., individual packets are successfully transmitted with finite delay almost surely ††)) if

$$\lambda < \lambda_{crit} - \varepsilon_\ell \quad . \qquad (3.41)$$

Conversely, a sufficient condition for instability is

$$\lambda > \lambda_{crit} + \varepsilon_\ell \quad . \qquad (3.42)$$

In equations (3.41) and (3.42) we used the definitions

$$\lambda_{crit} \triangleq (\overline{\alpha_a})^{-1} \quad \text{and} \quad \varepsilon_{\ell,u} \approx \frac{\max_N |\alpha_a(N) - \overline{\alpha_a}|}{\overline{\alpha_a}^2} \,, \qquad (3.43)$$

where $\overline{\alpha_a}$ is the mean value (wrt $\log_a(N)$, as $N \to \infty$) of

$$\alpha_a(N) \triangleq \frac{L_N}{N_{xmt}} = \frac{L_N}{N} \quad , \quad \text{as } N \to \infty \,, \qquad (3.44)$$

where N_{xmt} denotes the number of packets successfully

---

†) Actually, for the blocked access RAS's which we treat here, the critical measure for stability considerations in the above sense is the mean number of newly arriving packets per slot; in that case, the distribution of the number of new packets per slot is not essential, provided that the new packet process is sufficiently time homogeneous and independent of the state of the system, see [FAY-HOF83] or [HOF82].

††) The qualifier almost surely is necessary since it is possible (with vanishing probability, however) that two or more transmitters flip exactly the same "coin" sequence.

transmitted during L_N. In the case of Poisson arrivals, our conjecture is that (with the above definition for stability) λ_{crit} is exactly the limit of stability; we have, however, not been able to prove this so far when fair coins are utilized (cf. comment (8)). For a proof of (3.41) and (3.42), see [FAY-HOF83].

(3) Equation (3.40) can also be obtained in closed form for arbitrarily biased coins, see [MAT84] and, for a case with Q=2, [HOF82]. A third method to solve (3.20) (and (3.27)) is to replace $L^*(z)$ by $L^*(z)-1$. The resulting equation can then easily be solved by Mellin transform techniques. We do not further investigate this technique here since it can not be generalized for the free access RAS's.

<u>Asymptotic analysis</u>. From comment (2) we see that one needs the quantities $\alpha_a(N)$ and $\overline{\alpha_a}$ so as to be able to say anything about the stability of the blocked access RAS's. To compute $\alpha_a(N)$ for large N we use the exponential approximation

$$(1-Q^{-m})^N = e^{-N\cdot Q^{-m} + (N\cdot Q^{-m})^2\cdot O(N^{-1})} \quad ^{+)} \qquad . \qquad (3.45)$$

This suggests approximating (3.40) by

+) We say that f(n)=O(g(n)) if one can find integers N and M such that |f(n)|≤M·g(n) for all n≥N, see for instance [GRE-KNU82].

$$\breve{L}_N = 1 + Q\cdot\sum_{m=0}^{\infty} Q^m\cdot(1-e^{-N\cdot Q^{-m}} - N\cdot Q^{-m}\cdot e^{-N\cdot Q^{-m}}) +$$
$$- \delta_N\cdot\sum_{m=0}^{\infty} Q^m\cdot(e^{-N\cdot Q^{-m}\cdot(1-Q^{-1})} - e^{-N\cdot Q^{-m}} - N\cdot Q^{-m-1}\cdot e^{-N\cdot Q^{-m}}) . \tag{3.46}$$

It can be shown that $L_N-\breve{L}_N=O(1)$ as $N\to\infty$, see [MAT84] and, for a similar case in the context of radix exchange sorting, [KNU73, pp. 131-132].

To decompose (3.46) and isolate N one can use the Mellin transform and the Mellin summation formula [DAV78], or, equivalently, use a relation of the gamma function to replace e^{-x}, see for example [KNU73, pp. 132-134]. The Mellin transform (if it exists) of some function f(x) is defined as [DOE50], [DAV78]

$$\mathcal{M}[f(x);s] = F(s) \triangleq \int_0^{\infty} x^{s-1}\cdot f(x)\cdot dx \quad , \qquad a < Re(s) < b \ , \tag{3.47}$$

with the corresponding inversion formula

$$f(x) \triangleq \frac{1}{2\pi i}\cdot\int_{c-i\infty}^{c+i\infty} x^{-s}\cdot F(s)\cdot ds \quad , \qquad a < c < b \ . \tag{3.48}$$

Letting $x_m=N\cdot Q^{-m}$, the Mellin transforms required for (3.46) are

$$\mathcal{M}[1-e^{-x_m}-x_m\cdot e^{-x_m};s] = -(s+1)\cdot\Gamma(s) \quad , \quad -2 < Re(s) < 0 \ , \tag{3.49}$$

$$\mathcal{M}[e^{-x_m\cdot(1-Q^{-1})} - e^{-x_m} - x_m\cdot Q^{-1}\cdot e^{-x_m};s] =$$
$$= -[1+Q^{-1}\cdot s-(1-Q^{-1})^{-s}]\cdot\Gamma(s) \quad , \quad Re(s) > -2. \tag{3.50}$$

Thus, applying the Mellin summation formula we may express (3.46) as

$$\tilde{L}_N = 1 - \frac{Q}{2\pi i} \cdot \int_{-3/2-i\infty}^{-3/2+i\infty} (s+1)\cdot\Gamma(s)\cdot\sum_{m=0}^{\infty} Q^m \cdot x_m^{-s}\cdot ds \; +$$

$$+ \frac{\delta_M}{2\pi i} \cdot \int_{-3/2-i\infty}^{-3/2+i\infty} [1+Q^{-1}\cdot s-(1-Q^{-1})^{-s}]\cdot\Gamma(s)\cdot\sum_{m=0}^{\infty} Q^m \cdot x_m^{-s}\cdot ds \;. \tag{3.51}$$

Equation (3.51) has the pleasing property that N and Q^{-m} can be separated, thereby allowing us to evaluate the sum over m. This leads to

$$\tilde{L}_N = 1 - \frac{Q}{2\pi i} \cdot \int_{-3/2-i\infty}^{-3/2+i\infty} \frac{(s+1)\cdot\Gamma(s)\cdot N^{-s}}{1-Q^{s+1}}\cdot ds \; +$$

$$+ \frac{\delta_M}{2\pi i} \cdot \int_{-3/2-i\infty}^{-3/2+i\infty} \frac{[1+Q^{-1}\cdot s-(1-Q^{-1})^{-s}]\cdot\Gamma(s)\cdot N^{-s}}{1-Q^{s+1}}\cdot ds \;. \tag{3.52}$$

The line integrals in (3.52) can be evaluated by using the residue theorem (cf. Fig. 3.2). If we close the contour of the inversion integrals in the left hand half plane we obtain an ascending (in terms of powers of N) expansion which is just equal to (3.46). We are, however, interested in the asymptotic behaviour of $\tilde{L}_N$ as $N \to \infty$ and therefore we close the path of integration in the right hand half plane (descending expansion). Hence, we take the negative of the sum of

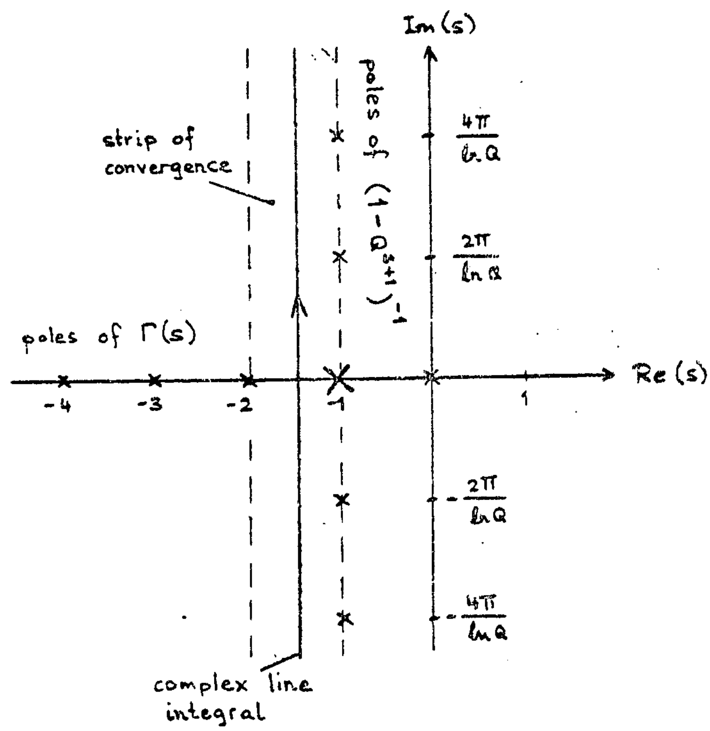

Fig. 3.2 Location of poles of the integrands in equation (3.52)

the residues of the poles to the right of -3/2 which yields for N>>1

$$\tilde{L}_N = \frac{N}{\ln Q}\cdot\{Q\cdot[1-f_1(N)] - \delta_M\cdot[Q^{-1}+(1-Q^{-1})\cdot\ln(1-Q^{-1})-f_2(N)]\} + $$
$$- \frac{1}{Q-1} + O(N^{-1}) \quad , \qquad (3.53)$$

where $f_1(N)$ and $f_2(N)$ are Fourier series (with small amplitude) in $\log_Q(N)$ of the form

$$f_1(N) = \sum_{k\neq 0} \frac{2\pi i k}{\ln Q}\cdot\Gamma(-1+\frac{2\pi i k}{\ln Q})\cdot e^{-2\pi i k\cdot\log_Q(N)} \quad , \qquad (3.54)$$

$$f_2(N) = \sum_{k\neq 0} [(1-Q^{-1})\cdot(1-e^{-2\pi i k\cdot\log_Q(1-Q^{-1})}) + \frac{2\pi i k}{Q\cdot\ln Q}] \times$$
$$\times\ \Gamma(-1+\frac{2\pi i k}{\ln Q})\cdot e^{-2\pi i k\cdot\log_Q(N)} \quad , \qquad (3.55)$$

with $f(N)\equiv f(Q\cdot N)$.

For the quantities $\alpha_Q(N)$ and $\overline{\alpha_Q}$ in which we are primarily interested we get for $N\to\infty$ (using (3.44) with $L_N \approx \tilde{L}_N$)

$$\alpha_Q(N) = \frac{Q\cdot[1-f_1(N)] - \delta_M\cdot[Q^{-1}+(1-Q^{-1})\cdot\ln(1-Q^{-1}) - f_2(N)]}{\ln Q} \quad , \qquad (3.56)$$

and, when neglecting the fluctuating terms $f_1(N)$ and $f_2(N)$

$$\overline{\alpha_Q} = \frac{Q - \delta_M\cdot[Q^{-1}+(1-Q^{-1})\cdot\ln(1-Q^{-1})]}{\ln Q} \quad . \qquad (3.57)$$

Comments:

(4) The use of the residue theorem to obtain (3.53) needs some justifications which we tacitly omitted. The interested reader may find these in [MAT84] and [KNU73,

pp. 132-133] or, in more general form, in [DOE55]. For a tutorial on the use of Mellin transform techniques for the analysis of computer algorithms we refer to [FLA-REG-SED84].

(5) The fact that a quantity like $\alpha_a(N)$ does not necessarily have a limit as $N \to \infty$ was first shown in [KNU73, p. 134] in the context of radix exchange sorting. For the blocked access RAS's with Q=2 and fair coins this was first proved in [HAJ80] by the use of the well-known approximation of binomial probabilities by Poisson probabilities. For biased coins see comment (8) below.

(6) The same techniques that led to (3.53) can be used to obtain an asymptotic expression for S_N and, using the relation

$$Var_N = S_N - L_N \quad , \tag{3.58}$$

for the asymptotic behaviour of Var_N (the conditional variance of the CRI-length given N packets initially collided). In particular, it can be shown that Var_N is linear in N as $N \to \infty$. The derivation of these two quantities, however, is rather lengthy and complicated (cf. [MAT84]) and is therefore omitted in this paper.

(7) Another technique to obtain arbitrarily tight bounds on $\alpha_a(N)$ and, with some restrictions, on Var_N/N for $N \to \infty$ uses an inductive argument for all N greater

than some fixed threshold M, for CRA's with Q=2 see [AMA81] or [MAS81].

(8) Biased coins. The expression corresponding to (3.52) above is

$$\tilde{L}_N = 1 - \frac{Q}{2\pi i} \cdot \int_{-3/2-i\infty}^{-3/2+i\infty} \frac{(s+1)\cdot\Gamma(s)\cdot N^{-s}}{1-\sum_{j=1}^{Q} p_j^{-s}} \cdot ds \; +$$

$$+ \frac{\delta_M}{2\pi i} \cdot \int_{-3/2-i\infty}^{-3/2+i\infty} \frac{[1+p_Q\cdot s-(1-p_Q)^{-s}]\cdot\Gamma(s)\cdot N^{-s}}{1-\sum_{j=1}^{Q} p_j^{-s}} \cdot ds \;. \quad (3.59)$$

Again, the asymptotic behaviour of $\tilde{L}_N$ as $N \to \infty$ is determined by the poles to the right of -3/2. In particular, the fluctuations of $\tilde{L}_N$ are determined by the poles at $s=x\pm iy$, where x and y are the solutions of

$$\sum_{j=1}^{Q} p_j^{-x} \cdot \cos(y \cdot \ln p_j) = 1, \quad \sum_{j=1}^{Q} p_j^{-x} \cdot \sin(y \cdot \ln p_j) = 0,$$

$$x \geqslant -1 \;,\; y > 0 \;. \quad (3.60)$$

Letting x=1 in (3.60) we see that since $\sum_{j=1}^{Q} p_j = 1$ solutions for y>0 are possible if and only if we can find positive integers k_j such that

$$p_1^{1/k_1} = p_2^{1/k_2} = \ldots = p_Q^{1/k_Q} \;. \quad (3.61)$$

Consequently, the amplitude of the fluctuations of $\tilde{L}_N$ is proportional to N iff (3.61) is satisfied; otherwise the amplitude is proportional to $N^{1-\delta}$ where δ is some (small) positive constant. In the latter case we find that $\lim_{N\to\infty} \alpha_Q(N)=\overline{\alpha_Q}$ whence we have the result that the blocked access RAS's will be stable for all arrival rates $\lambda < \lambda_{crit} \triangleq (\overline{\alpha_Q})^{-1}$ if biased coins are used such that (3.61) is not satisfied. Note that if one were able to show that the maximum stable throughput of the blocked access RAS's must be a continuous function of the biasing of the coins then our conjecture in comment (2) would hold. For $\overline{\alpha_Q}$ in the biased coin case we obtain

$$\overline{\alpha_Q} = \frac{Q - \delta_M \cdot [p_Q + (1-p_Q) \cdot \ln(1-p_Q)]}{-\sum_{j=1}^{Q} p_j \cdot \ln p_j} \quad . \tag{3.62}$$

Numerical results. Here we give the numerical values of λ_{crit} and ε_ℓ for Q=2..10. Numerical values for $\overline{\alpha_Q}$,

Q	Basic CRA λ_{crit}	Basic CRA ε_ℓ	Modified CRA λ_{crit}	Modified CRA ε_ℓ
2	.346574	.00000038	.375369	.00000033
3	.366204	.0000948	.374062	.0000910
4	.346574	.000644	.349566	.000641
5	.321888	.001712	.323277	.001712
6	.298627	.003093	.299362	.003095
7	.277987	.004590	.278414	.004593
8	.259930	.006075	.260196	.006078
9	.244136	.007479	.244310	.007483
10	.230259	.008771	.230378	.008775

Table 3.1 λ_{crit} and ε_ℓ as a function of Q for the blocked access RAS's with fair coins (The systems are stable for all $\lambda < \lambda_{crit} - \varepsilon_\ell$).

$\max_N |\alpha_Q(N) - \overline{\alpha_Q}|$, L_N, S_N and Var_N can be found in Appendix C. All computations were done on an 8-bit microprocessor development system using the equivalent of Fortran double precision arithmetic. Numerical stability proved to be satisfactory for moderate N (up to N≈30) with the direct meth-

Q	Modified CRA with biased coins	
	optimum p_a	λ_{crit}
2	.582492	.381260
3	.373358	.375087
4	.271332	.349834

Table 3.2. λ_{crit} and p_a for the modified blocked access RAS's with optimally biased coins. The coin values 1,2,...,Q-1 are equally likely and have probability p, with $p=(1-p_a)/(Q-1)$.

od; with the iteration method and suitable regroupings (cf. Appendix D) numerical stability is excellent, even for N>10'000. The results for Q=2 are not new, they merely serve as a reference for comparisons.

Comments:

(9) The basic CRA performs optimally (in terms of λ_{crit}) when fair coins are utilized (cf. (3.62)). Intuitively, this can be estimated from the symmetry of the respective equations with respect to the p_j. The optimum, however, is quite flat; slightly biased coins do not substantially affect system performance. The modified CRA, on the other hand, performs somewhat better for p_a greater than Q^{-1} (cf. table 3.2).

(10) The modified CRA with Q=2 is slightly superior to the same CRA with Q=3. This superiority is further increased when biased coins are used (cf. table 3.2). It should be noted, however, that the modified blocked access RAS's can suffer from deadlocks in the presence

of channel errors (cf. [MAS81]).

(11) Note that for N less than 4 the CRA with Q=2 is generally optimum for all blocked access RAS's (cf. tables C.3...C.6 in Appendix C); this is essentially what Capetanakis proved for his "dynamic tree algorithm" [CAP77]. All of the non-obvious blocked access RAS's known make use of this property of the CRA with Q=2 to increase the system throughput to values above those given in this paper; at the expense, however, of increased system complexity and stronger requirements on the distribution of the new packet arrival process. At present, the best value known is (using a modified CRA with Q=2) λ_{crit} =.48775, see [HUM-MOS80] and [TSY-MIK80].

IV. Analysis of RAS's with free access and basic CRA

With the techniques in mind which we developed in the previous section we are now ready to attack the practically more interesting (but mathematically more complex) free access RAS's. In this section (which will be concerned with the basic CRA case only) we develop first the tgf for the L_N. We then solve $L^*(z)$ (for the sake of simplicity for fair coins only) by the iteration method and show that the crucial quantity to determine the limit of the maximum stable throughput (λ_{crit}) of the free access RAS's is $L^*(\lambda)$. This means that in contrast to the previous section an asymptotic analysis of L_N as $N \longrightarrow \infty$ is not required to determine λ_{crit}. We will, however, conclude this section with an asymptotic expression for L_N as $N \rightarrow \infty$ since this result is of some interest in its own. For the solution of $L^*(z)$ by the direct method (for arbitrarily biased coins) we refer to Section V.

The starting point for the analysis of the basic free access RAS's is again the inspection of one stage of the splitting process (see Fig. 4.1). X_j denotes the (random) number of newly arriving packets in the root node of the j-th subtree. From Fig. 4.1 we im-

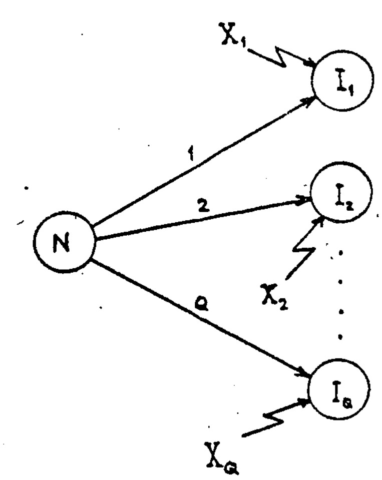

Fig. 4.1 Splitting N packets to Q subsets for the basic free access RAS's.

mediately obtain the recurrence

$$Y_N = 1 + \sum_{j=1}^{Q} Y_{I_j + X_j} \quad , \quad N \geq 2 \quad , \qquad (4.1)$$

with initial values as in (3.3). Using the same definitions and procedures as in Section III (this time conditioning on I_j and X_j) and making use of the assumptions (i) the new packet arrival process is independent of the collision resolution process, (ii) the new packet arrival processes of the Q subtrees are mutually indpendent and (iii) the number of transmitters is infinite, we obtain[+)]

$$G_N(s) = s \cdot \sum_{i_1 \ldots i_Q} \binom{N}{i_1 \ldots i_Q} \cdot \prod_{j=1}^{Q} p_j^{i_j} \cdot \sum_{x_j=0}^{\infty} p(x_j) \cdot G_{i_j + x_j}(s) \quad , \quad N \geq 2, \qquad (4.2)$$

with initial conditions (3.10) and definitions (3.7)... ..(3.9). $p(x_j)$ denotes the probability distribution of the new arrivals in the root node of the j-th subtree and, assuming a Poisson arrival process with rate λ (packets/slot), we have

$$p(x_j) = Pr\{X_j = x_j\} = e^{-\lambda} \cdot \frac{\lambda^{x_j}}{x_j!} \quad . \qquad (4.3)$$

Substituting (4.3) in (4.2), taking the first derivative of the resulting expression wrt s and evaluating it at s=1 therefore yields

$$L_N = 1 + \sum_{j=1}^{Q} \sum_{i_j=0}^{N} \binom{N}{i_j} \cdot p_j^{i_j} \cdot (1-p_j)^{N-i_j} \cdot e^{-\lambda} \cdot \sum_{x_j=0}^{\infty} L_{i_j + x_j} \cdot \frac{\lambda^{x_j}}{x_j!} \quad ,$$

$$N \geq 2 \quad , \qquad (4.4)$$

+) Assumption (iii) is essential because otherwise the summation over x_j in eq. (4.2) would depend on N and this changes the nature of (4.2) and thereby the RAS which it describes essentially, see [MAT84].

with initial conditions (3.14). Inspecting the last equation we see that unlike for the blocked access RAS's, L_N can not be computed recursively anymore for the free access RAS's.

Indeed, to elicit any information from (4.4) one has to resort to techniques like using some form of truncation (like the one used in [TSY-VVE80]) or like introducing generating functions for L_N. Noting that

$$L^{(1)}(z) = \sum_{N=0}^{\infty} L_{N+1} \cdot \frac{z^N}{N!} \quad , \text{ and} \tag{4.5}$$

$$L^{*(1)}(z) = e^{-z} \cdot [L^{(1)}(z) - L(z)] = \sum_{k=1}^{\infty} k \cdot L_k^* \cdot z^{k-1} \quad , \tag{4.6}$$

we proceed directly to the tgf of L_N (using the same procedures and definitions as we did to obtain (3.20)) and thus

$$L^*(z) - \sum_{j=1}^{Q} L^*(\lambda + p_j \cdot z) = 1 + Q \cdot L^*(\lambda) \cdot f^*(z) + L^{*(1)}(\lambda) \cdot g^*(z) \quad , \tag{4.7}$$

where $f^*(z) = - e^{-z} \cdot (1+z)$, (4.8)

and $g^*(z) = - z \cdot e^{-z}$, (4.9)

with initial conditions (3.23). Note that the RHS of (4.7) contains two unknown constants (constant wrt z), namely $L^*(\lambda)$ and $L^{*(1)}(\lambda)$. The tgf for the S_N can be obtained in the same manner as (3.27) was obtained, see Appendix B for the result.

For reasons of simplicity, we again discuss the iteration method to solve (4.7) only for fair coins. Differentiating (4.7) for fair coins twice wrt z yields

$$L^{*(2)}(z)-Q^{-1}\cdot L^{*(2)}(\lambda+Q^{-1}\cdot z) = Q\cdot L^{*}(\lambda)\cdot f^{*(2)}(z)+L^{*(1)}(\lambda)\cdot g^{*(2)}(z), \tag{4.10}$$

which satisfies the contraction condition (cf. Appendix A). Noting that this time

$$\sigma_M^{[m]}(z) = \sigma_M^{[m]}(0) + Q^{-m}\cdot z = \mu\cdot(1-Q^{-m}) + Q^{-m}\cdot z \quad , \tag{4.11}$$

where $\mu \triangleq \dfrac{\lambda}{1-Q^{-1}}$, (4.12)

equation (4.10) has the solution

$$L^{*(2)}(z)=Q\cdot L^{*}(\lambda)\cdot\sum_{m=0}^{\infty} Q^{-m}\cdot f^{*(2)}(\sigma_M^{[m]}(z))+L^{*(1)}(\lambda)\cdot\sum_{m=0}^{\infty} Q^{-m}\cdot g^{*(2)}(\sigma_M^{[m]}(z)). \tag{4.13}$$

As in Section III, we have to integrate (4.13) twice to recover $L^{*}(z)$. In the course of carrying out these integrations we will also get, almost as a byproduct, the solutions for the two unknown constants $L^{*(1)}(\lambda)$ and $L^{*}(\lambda)$. Using (3.23) and definition (3.35), the first integration amounts to

$$L^{*(1)}(z) = Q\cdot L^{*}(\lambda)\cdot \mathcal{R}^{(1)}(f^{*}(.);z) + L^{*(1)}(\lambda)\cdot \mathcal{R}^{(1)}(g^{*}(.);z) \quad , \tag{4.14}$$

whence letting $z=\lambda$ and solving for $L^{*(1)}(\lambda)$ we obtain

$$L^{*(1)}(\lambda) = Q\cdot L^{*}(\lambda)\cdot \frac{\mathcal{R}^{(1)}(f^{*}(.);\lambda)}{1-\mathcal{R}^{(1)}(g^{*}(.);\lambda)} = Q\cdot L^{*}(\lambda)\cdot\frac{\mu}{1-\mu} . \tag{4.15}$$

The last equality in (4.15) follows from observing that

$$\sigma_M^{[m]}(\lambda) = \lambda\cdot\frac{1-Q^{-m}}{1-Q^{-1}} + \lambda\cdot Q^{-m} = \sigma_M^{[m+1]}(0) \quad , \tag{4.16}$$

and hence

$$\mathfrak{R}^{(1)}(\Psi(.);\lambda) = \sum_{m=0}^{\infty} [\Psi^{(1)}(\sigma_M^{[m+1]}(0)) - \Psi^{(1)}(\sigma_M^{[m]}(0))]$$

$$= \Psi^{(1)}(\mu) - \Psi^{(1)}(0) \quad . \qquad (4.17)$$

Integrating (4.14) brings us back to $L^*(z)$ which may now be written as (with (4.15) and (3.23) substituted and using definition (3.36))

$$L^*(z) = 1 + Q \cdot L^*(\lambda) \cdot d^*(z) \quad , \qquad (4.18)$$

where $d^*(z) \triangleq \mathfrak{R}(f^*(.);z) + \mu \cdot K_L(\lambda) \cdot \mathfrak{R}(g^*(.);z)$, (4.19)

and $K_L(\lambda) \triangleq 1/(1-\mu)$. (4.20)

Note that with (4.12) substituted the condition for $K_L(\lambda)$ to be positive and finite is

$$\lambda < (Q-1)/Q \quad . \qquad (4.21)$$

The next step is to get rid of the last unknown on the RHS of (4.18), i.e., to determine $L^*(\lambda)$. To this end, let $z=\lambda$ in (4.18) and solve for $L^*(\lambda)$ which yields

$$L^*(\lambda) = \frac{1}{1-Q\cdot d^*(\lambda)} \quad , \qquad (4.22)$$

with

$$d^*(\lambda) = K_L(\lambda) \cdot e^{-\mu} \cdot \sum_{m=0}^{\infty} Q^m \cdot \{[1-\mu \cdot Q^{-m} + \mu^2 \cdot Q^{-2m} \cdot (1-Q^{-1})] \cdot e^{\mu \cdot Q^{-m}} +$$

$$- (1-\mu \cdot Q^{-m-1}) \cdot e^{\mu \cdot Q^{-m-1}}\} \quad . \qquad (4.23)$$

The crucial quantity which determines λ_{crit} is readily seen to be the denominator on the RHS of (4.22). Hence, we define λ_{crit} implicitly as

$$d^*(\lambda_{crit}) \triangleq Q^{-1} \quad , \quad Q > 1 \quad , \qquad (4.24)$$

where λ_{crit} has to satisfy (4.21) in order to guarantee that definition (4.24) is unique.

Returning to equation (4.18), we obtain, upon multiplying both sides by e^z and equating coefficients of $z^N/N!$, an explicit expression for L_N, i.e.,

$$L_N = 1 + Q \cdot L^*(\lambda) \cdot d_N \qquad , \qquad N \geqslant 2 \quad , \tag{4.25}$$

with $$d_N = K_L(\lambda) \cdot e^{-\mu} \cdot \{\mu \cdot \sum_{m=0}^{\infty} e^{\mu \cdot Q^{-m}} \cdot [(1-Q^{-m})^N + N \cdot Q^{-m} - 1] + \sum_{m=0}^{\infty} Q^m \cdot e^{\mu \cdot Q^{-m}} \cdot [1-(1-Q^{-m})^N - N \cdot Q^{-m} \cdot (1-Q^{-m})^{N-1}]\} \;, \quad N \geqslant 2 \;. \tag{4.26}$$

The rather messy expression for S_N can be derived in the same manner, see Appendix B for the final equation.

Comments:

(1) From equation (4.22) we see that (since L_N must be positive and λ must be greater or equal to zero and satisfy (4.21)) $L^*(\lambda)$ and therefore (by (4.25)) the first moment of Y_N exist if and only if

$$\lambda < \lambda_{crit} \qquad \text{and} \qquad Q > 1 \qquad . \tag{4.27}$$

It can be shown that the above conditions not only guarantee the existence of L_N but also the existence of all higher moments of Y_N (cf. Appendix B for the second moment). For $\lambda \geqslant \lambda_{crit}$, the functional equation (4.7) cannot provide a solution for equation (4.4). Inspecting (4.4) we see, however, that the only solution in this latter case is $L_N = \infty$ for $N \geqslant 2$.

(2) From comment (1) above it is clear that the basic free access RAS's are stable (cf. comment (2), Section III) for Poisson arrivals with rate λ if and and only if λ and Q satisfy (4.27). Note that, in contrast to the blocked access RAS's, the turning point between stability and instability can be determined exactly from (4.24) without the need of an asymptotic analysis. For comparison purposes we shall, nevertheless, investigate the behaviour of L_N/N as $N \to \infty$ and we define

$$\beta_a(N) \triangleq \frac{L_N}{N_{coll}} = \frac{L_N}{N} \quad , \quad \text{as } N \to \infty \quad , \qquad (4.28)$$

where N_{coll} denotes the number of packets which initially collided. Since we are dealing with free access RAS's, N_{xmt} (cf. comment (2), Section III) is greater than N_{coll} for $N_{coll} \geq 2$ and $\lambda > 0$, namely

$$N_{xmt} = N_{coll} + \lambda \cdot (L_N - 1) \quad . \qquad (4.29)$$

We can therefore establish the following relation between $\beta_a(N)$ and $\alpha_a(N)$ (as defined in (3.44))

$$\alpha_a(N) = \frac{L_N}{N_{coll} + \lambda \cdot (L_N - 1)} = \frac{\beta_a(N)}{1 + \lambda \cdot \beta_a(N)} \quad , \quad \text{as } N \to \infty . \qquad (4.30)$$

Again, $\overline{\alpha_a}$ and $\overline{\beta_a}$ denote the mean values (wrt log (N), as $N \to \infty$) of $\alpha_a(N)$ and $\beta_a(N)$, respectively.

(3) A convenient consequence (which is at first not obvious) of our using tgf's is that for the free access RAS's with Poisson arrivals quantities like $L^*(\lambda)$ and $S^*(\lambda)$ (cf. Appendix B) admit a physical interpreta-

tion, viz. as the unconditional first and second moment of the CRI-length. To see this, we first note that for the free access RAS's the transmission process has recurrent renewal points which are best described by the state of the system stack. One such set of renewal points are the time instants at which the system stack is empty (i.e., the time instants between disjoint CRI's). The interpretation of $L^*(\lambda)$ and $S^*(\lambda)$ follows then immediatley by making use of the Poisson arrival process assumption and by evaluating equations (3.18) and (3.26) (with (3.15) and (3.25), respectively, substituted) at $z=\lambda$. Hence, we may write

$$E[Y] \equiv L^*(\lambda) \quad \text{and} \quad E[Y^2] \equiv S^*(\lambda) \quad . \qquad (4.31)$$

For the unconditional variance of the CRI-length we have thus

$$Var[Y] = S^*(\lambda) - [L^*(\lambda)]^2 \quad . \qquad (4.32)$$

(4) For $Q=2$ and arbitrarily biased coins, the quantities $L^{*(1)}(\lambda)$ (cf. (4.15)) and $S^{*(1)}(\lambda)$ (cf. Appendix B) can also be determined (in terms of $L^*(\lambda)$ and $S^*(\lambda)$) by evaluating equations (4.7) and (B.7), respectively, at the fixed points $z_1=\lambda+p_1 \cdot z_1$ and $z_2=\lambda+p_2 \cdot z_2$ (i.e., $z_1 = \lambda/(1-p_1)$, $z_2=\lambda/(1-p_2)$), see [FAY-FLA-HOF-JAC83]. For $Q>2$ this technique can only be used for fair coins, in which case one has to evaluate the first derivative wrt z of (4.7) and (B.7), respectively, at $z_0=\lambda+Q^{-1} \cdot z_0$ (i.e., $z_0=\mu$).

Asymptotic analysis. To compute the quantity $\beta_a(N)$ which we introduced in comment (2) above we again make use of approximation (3.45) to obtain (from (4.25))

$$\tilde{L}_N = 1 + Q\cdot L^*(\lambda)\cdot K_L(\lambda)\cdot e^{-\mu}\cdot [\mu\cdot\sum_{m=0}^{\infty} e^{\mu\cdot Q^{-m}}\cdot(e^{-N\cdot Q^{-m}} + N\cdot Q^{-m} - 1) +$$

$$+ \sum_{m=0}^{\infty} Q^m\cdot e^{\mu\cdot Q^{-m}}\cdot(1 - e^{-N\cdot Q^{-m}} - N\cdot Q^{-m}\cdot e^{-N\cdot Q^{-m}})] \quad . \qquad (4.33)$$

Note that $L_N - \tilde{L}_N = O(1)$ as $N \longrightarrow \infty$ (cf. Section III). To isolate N we again have recourse to Mellin transform techniques. The same steps that led to (3.52) yield for the basic free access RAS's

$$\tilde{L}_N = 1 + Q\cdot L^*(\lambda)\cdot K_L(\lambda)\cdot e^{-\mu}\cdot [\mu\cdot\sum_{\ell=0}^{\infty}\frac{\mu^\ell}{\ell!}\cdot\frac{1}{2\pi i}\cdot\int_{-3/2-i\infty}^{-3/2+i\infty}\frac{\Gamma(s)\cdot N^{-s}}{1-Q^{s-\ell}}\cdot ds +$$

$$-\sum_{\ell=0}^{\infty}\frac{\mu^\ell}{\ell!}\cdot\frac{1}{2\pi i}\cdot\int_{-3/2-i\infty}^{-3/2+i\infty}\frac{(s+1)\cdot\Gamma(s)\cdot N^{-s}}{1-Q^{s+1-\ell}}\cdot ds] \quad . \qquad (4.34)$$

The descending expansion (poles to the right of -3/2) yields in this case

$$\tilde{L}_N = Q\cdot L^*(\lambda)\cdot K_L(\lambda)\cdot e^{-\mu}\cdot N\cdot [\sum_{m=0}^{\infty}\mu\cdot Q^{-m}\cdot e^{\mu\cdot Q^{-m}} + \frac{1}{\ln Q} - \frac{f_1(N)}{\ln Q}] + O(1) \, , \qquad (4.35)$$

with $f_1(N)$ as defined in (3.54). Hence, neglecting the small fluctuations $f_1(N)$, setting $L_N \approx \tilde{L}_N$ and using (4.28) we obtain for $N \longrightarrow \infty$

$$\bar{\beta}_a = Q\cdot L^*(\lambda)\cdot K_L(\lambda)\cdot e^{-\mu}\cdot(\sum_{m=0}^{\infty}\mu\cdot Q^{-m}\cdot e^{\mu\cdot Q^{-m}} + \frac{1}{\ln Q}) \quad . \qquad (4.36)$$

Comments:

(5) As can be seen by a direct comparison of (3.46) and (4.33), the first sum of the former is almost identical with the second sum of the latter. Consequently, $\lim_{N\to\infty} \beta_Q(N)$ does not exist for fair coins. Notice, however, that the limit of the first sum in (4.33) (when divided by N and letting $N\to\infty$) exists. This is due to the absence of the factor Q^m in this sum. We can therefore state that the fluctuating behaviour of $\beta_Q(N)$ is induced entirely by the underlying splitting process of the CRA and not by the new arrivals which transmit for the first time.

(6) Biased coins. As mentioned in Appendix A, the iteration method can be extended to the use of several, different Moebius transforms which is necessary for treating the biased coin cases analytically. The resulting expressions, however, turn out to be considerably more difficult than the ones we gave above for the utilization of fair coins. For small values of λ we obtained for the quantity $L^*(\lambda)$ (cf. [FAY-FLA-HOF-JAC83], Section 6.1, for the method used)

$$L^*(\lambda) = 1 + \frac{Q}{1-\Sigma p^2}\cdot\frac{\lambda^2}{2} + \frac{Q}{1-\Sigma p^2}\cdot\left(\frac{3}{1-\Sigma p^2} - \frac{1+2\cdot\Sigma p^2}{1-\Sigma p^3}\right)\cdot\frac{\lambda^3}{3} + O(\lambda^4) \quad , \qquad (4.37)$$

with $$\Sigma p^k \triangleq \sum_{j=1}^{Q} p_j^k \quad . \qquad (4.38)$$

From the symmetry of equations (4.7) and (4.37) with respect to the p_j one expects the use of fair coins to be optimum for the basic free access RAS's. This anticipation is confirmed by numerical results which we obtained with the direct method (cf. Section V), see Fig. 4.2 .

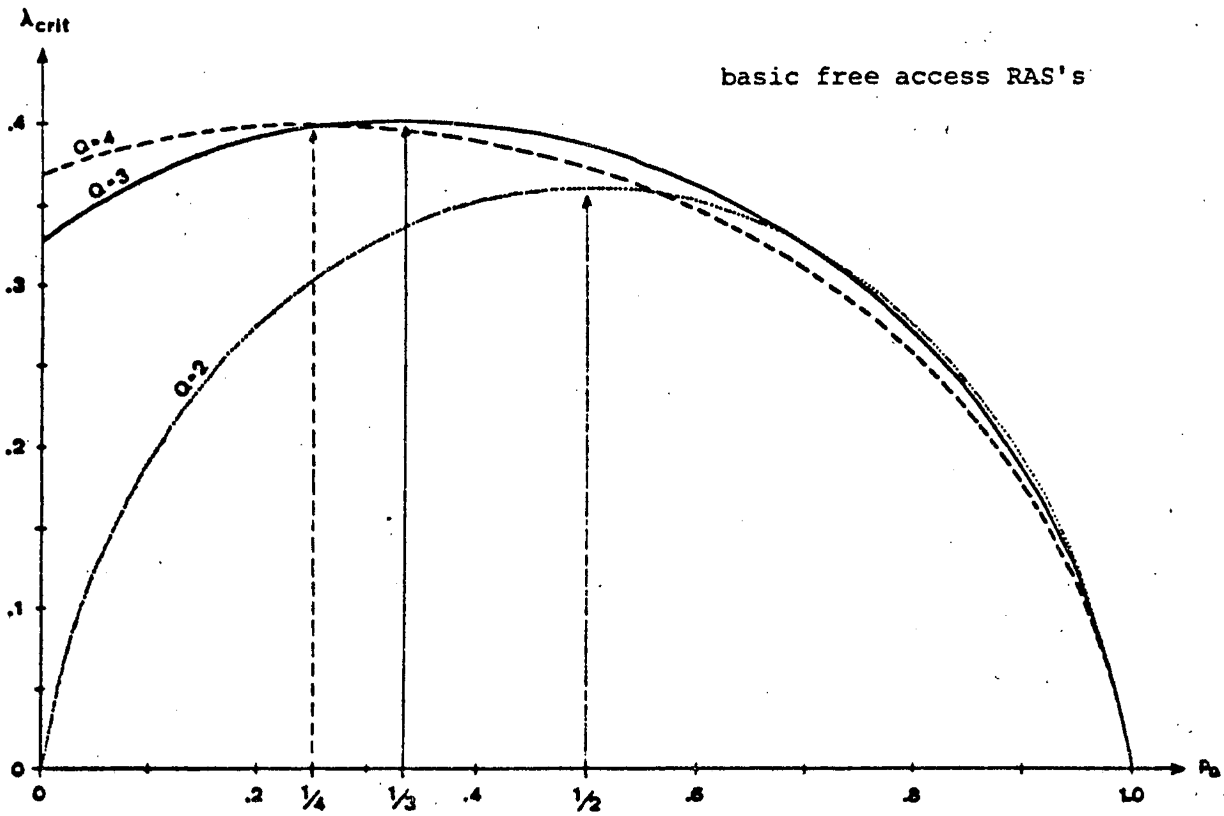

Fig. 4.2 Biased coins: λ_{crit} as a function of p_a, the probability that the value Q is flipped. All other values (1..Q-1) appear with probability p, where $p=(1-p_a)/(Q-1)$.

The asymptotic behaviour of $\widetilde{L}_N$ as $N\to\infty$ exhibits similar (but more involved) properties when biased coins are used as the ones we described in Section III, comment (8). In particular, we have that $\lim_{N\to\infty} \beta_a(N)=\overline{\beta_a}$

iff condition (3.61) is not satisfied by the p_j of the biased coins (cf. [FAY-FLA-HOF82] for Q=2).

Numerical results. Here we give the numerical values of λ_{crit} for Q=2..10. Numerical values for the other quantities which we introduced in this section are relegated to Appendix C. Computations were done by the same means as de-

Q	λ_{crit}
2	.360177
3	.401599
4	.399223
5	.387241
6	.373354
7	.359731
8	.347002
9	.335304
10	.324604

Table 4.1 λ_{crit} as a function of Q for the basic free access RAS's with fair coins (the systems are stable for all $\lambda < \lambda_{crit}$).

scribed in Section III. Numerical stability, also for large N, is excellent with suitable regroupings of terms (cf. Appendix D). Some of the results for Q=2 can also be found in [FAY-HOF82] and [FAY-FLA-HOF-JAC83].

Comments:

(7) There is a fundamental difference between CRA's combined with blocked access and CRA's combined with free access which perhaps shows up best when comparing and interpreting numerical results. In the blocked access

case, the behaviour of the CRA itself is completely insensitive to the actual rate λ with which new packets arrive at the system. This insensitivity even holds for rates beyond λ_{crit} for which, of course, the transmission delays get excessive but without substantially affecting the system throughput. In the free access case, on the other hand, the CRA depends heavily on λ, especially for λ close to λ_{crit}. Care must be taken then, not to exceed λ_{crit} because otherwise the newly arriving packets, to which free access is granted, tend to clog the system which leads to a drop in system throughput. Hence, when interpreting the numerical results in Appendix C, keep in mind that the indication of the actual rate λ is essential for the free access RAS's.

(8) When using the basic free access RAS's, Q=3 is by far the best choice. For $\lambda \geqslant .2$, the expected length of the CRI, when N packets initially collide, is minimized for all $N \geqslant 2$ if Q=3. For arrival rates less than .2, Q=2 performs slightly better. However, if one plots $L^*(\lambda)$ (the unconditional mean of the CRI-length) versus Q for various rates λ it is evident that this superiority of Q=2 for low λ is quite inessential compared to the superiority of Q=3 for λ approaching λ_{crit}, see Fig 4.3. Asymptotically, the system with Q=3 needs in the average somewhat more than 2.54 slots

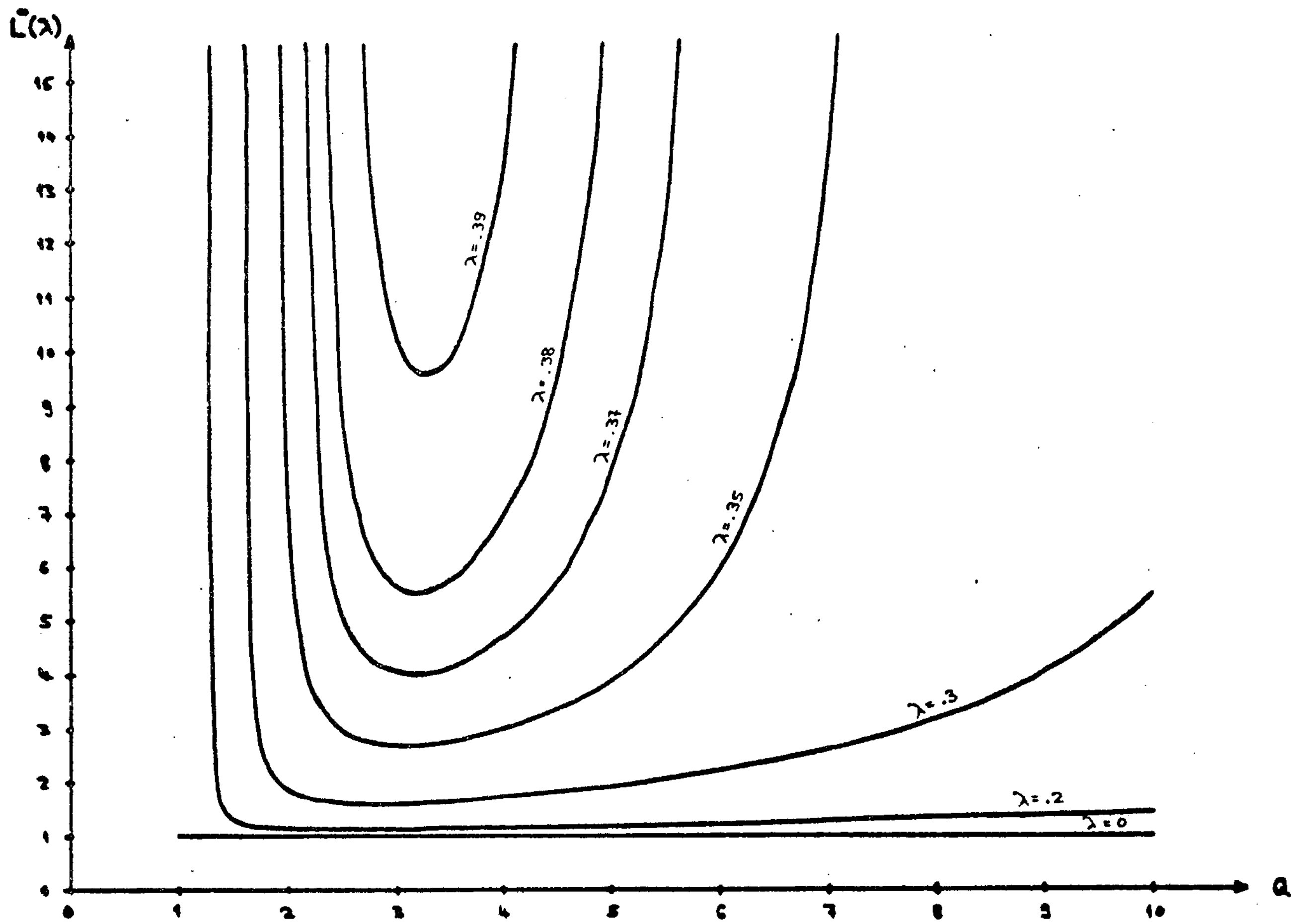

Fig. 4.3 Mean CRI-length ($L^*(\lambda)$) versus Q for the free access RAS's with fair coins at various arrival rates λ.

in order to successfully transmit one packet if $\lambda \approx .3$. The processing of packets speeds up to values slightly below 2.5 slots per successfully transmitted packet as λ reaches λ_{crit}.

V. Analysis of RAS's with free access and modified CRA

The main goal of this section is to provide a suitable method for computing the characteristic quantities of the modified free access RAS's. As a byproduct, this method will also yield an efficient way to produce numerical results for the basic free access RAS's when biased coins are used. First, we set up the tgf for the L_N, a procedure with which we are quite familiar by now. Then we use the direct method (cf. Section III) to obtain a solution in terms of the coefficients of the tgf. The crucial quantity which determines λ_{crit} and therefore the limit of stability of the system will again be $L^*(\lambda)$, the unconditional first moment of the CRI-length. The use of the direct method for the free access RAS's does not permit us to say anything about the asymptotic behaviour of L_N as $N \to \infty$ but, as we know from the previous section, this causes no hardship (in terms of stability considerations) in the free access case. We conclude this section by giving some hints how the iteration method can be extended to include the equations which one encounters for the modified free access RAS's.

We start by observing from Fig. 5.1 that the modification can only be enacted if after an initial collision all colliders flip the value Q and if at the same time no new packets arrive during Q-1 consecutive slots. For this

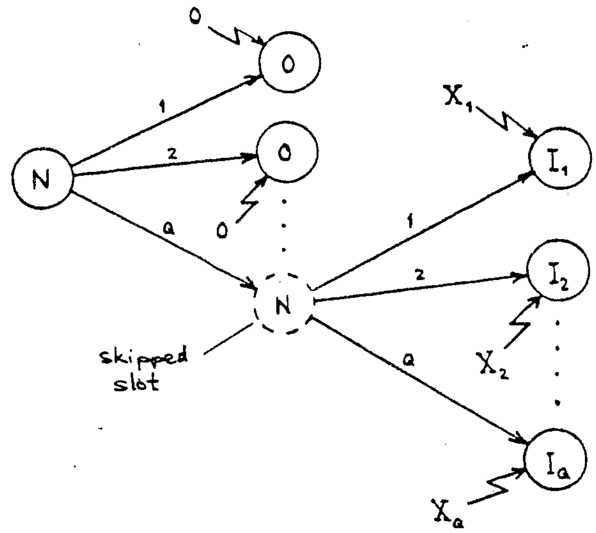

Fig. 5.1 Two stages of the splitting process in the modified case combined with free access.

event we obtain the recurrence

$$Y_N = 1 + (Q-1)\cdot Y_0 + (Y_N - 1) \qquad , \qquad N \geqslant 2 \quad . \qquad (5.1)$$

For all other cases the splitting process is described by recurrence (4.1). Note that the initial conditions for both recurrences are defined by (3.3). Let P_0 denote the probability of the event that no new packets are transmitted in the first Q-1 slots after a collision. Then we obtain, under the conditions stated for equation (4.2), in the same manner as we did for (3.11)

$$G_N(s) = s\cdot\sum_{i_1\ldots i_Q}^{N} \binom{N}{i_1\ldots\ldots i_Q}\cdot\prod_{j=1}^{Q} p_j^{i_j}\cdot\sum_{x_j=0}^{\infty} p(x_j)\cdot G_{i_j+x_j}(s) +$$

$$- \delta_M\cdot s^{Q-1}\cdot p_Q^N\cdot P_0\cdot[s\cdot\sum_{x_Q=0}^{\infty} p(x_Q)\cdot G_{N+x_Q}(s) - G_N(s)] \; ,$$

$$N \geqslant 2 \quad , \qquad (5.2)$$

with initial conditions (3.10) and definitions (3.7)...

..(3.9) and (3.12). Making use of the Poisson arrival process assumption whence we note that

$$P_o = e^{-\lambda\cdot(Q-1)} \quad , \tag{5.3}$$

we proceed directly to the tgf for the L_N which yields

$$L^*(z) - \sum_{j=1}^{Q} L^*(\lambda+p_j\cdot z) + \delta_M\cdot P_o\cdot e^{-z\cdot(1-p_Q)}\cdot[L^*(\lambda+p_Q\cdot z)-L^*(p_Q\cdot z)] =$$

$$= Q\cdot L^*(\lambda)\cdot f^*(z) + L^{*(1)}(\lambda)\cdot g^*(z) + h^*(z) \quad , \tag{5.4}$$

with $$f^*(z) = -e^{-z}\cdot[1+z-\delta_M\cdot P_o\cdot Q^{-1}\cdot(1+p_Q\cdot z)] \quad , \tag{5.5}$$

$$g^*(z) = -z\cdot e^{-z}\cdot(1-\delta_M\cdot P_o\cdot p_Q) \quad , \tag{5.6}$$

$$h^*(z) = 1-\delta_M\cdot P_o\cdot e^{-z\cdot(1-p_Q)} \quad , \tag{5.7}$$

and initial conditions (3.23). It is easily verified that for the basic free access RAS's (i.e., $\delta_M=0$) (5.4) reduces to (4.7) and for the blocked access RAS's (i.e., $\lambda=0$) (5.4) is equivalent to (3.20). For the tgf of the S_N, see Appendix B.

Applying the direct method to (5.4) is more complicated as opposed to its application in Section III which is due to the free access which is granted to new arrivals. Equating coefficients of z^k on both sides of (5.4) we get

$$L_k^*\cdot a_{kk} + \sum_{i=k+1}^{\infty} L_i^*\cdot a_{ik} + \delta_M\cdot\sum_{i=2}^{\infty} L_i^*\cdot b_{ik} =$$

$$= Q\cdot L^*(\lambda)\cdot f_k^* + L^{*(1)}(\lambda)\cdot g_k^* + h_k^* \quad , \quad k \geq 2 \quad , \tag{5.8}$$

with

$$a_{ik} = \begin{cases} 1 - \sum_{j=1}^{Q} p_j^k & , \quad i = k \geqslant 2 , \\ -\binom{i}{k} \cdot \lambda^{i-k} \cdot \sum_{j=1}^{Q} p_j^k & , \quad i > k \geqslant 2 , \end{cases} \tag{5.9}$$

$$b_{ik} = \begin{cases} P_o \cdot (p_a - 1)^k \cdot \sum_{l=0}^{i-1} \binom{i}{l} \cdot (\frac{p_a}{p_a - 1})^l \cdot \frac{\lambda^{i-l}}{(k-l)!} & , \quad k \geqslant i \geqslant 2 , \\ P_o \cdot (p_a - 1)^k \cdot \sum_{l=0}^{k} \binom{i}{l} \cdot (\frac{p_a}{p_a - 1})^l \cdot \frac{\lambda^{i-l}}{(k-l)!} & , \quad i > k \geqslant 2 , \end{cases} \tag{5.10}$$

and

$$f_k^* = \frac{(-1)^k}{k!} \cdot [k-1 - \delta_M \cdot P_o \cdot Q^{-1} \cdot (k \cdot p_a - 1)] \quad , \quad k \geqslant 2 \quad , \tag{5.11}$$

$$g_k^* = \frac{(-1)^k}{k!} \cdot k \cdot (1 - \delta_M \cdot P_o \cdot p_a) \quad , \quad k \geqslant 2 \quad , \tag{5.12}$$

$$h_k^* = - \delta_M \cdot P_o \cdot \frac{(p_a - 1)^k}{k!} \quad , \quad k \geqslant 2 \quad . \tag{5.13}$$

Equation (5.8) can be written as a linear system of infinite dimension

$$\vec{L^*} \cdot (A + \delta_M \cdot B) = Q \cdot L^*(\lambda) \cdot \vec{f^*} + L^{*(1)}(\lambda) \cdot \vec{g^*} + \vec{h^*} \quad , \tag{5.14}$$

where $\vec{L^*} \triangleq [L_2^*, L_3^*, L_4^*, \ldots\ldots]$, (5.15)

and $\vec{f^*}$, $\vec{g^*}$ and $\vec{h^*}$ are row vectors analogous to (5.15) with components as defined in (5.11)...(5.13), respectively. Matrix A in (5.14) is lower triangular with elements a_{ik} as defined in (5.9) whereas matrix B is fully populated with elements b_{ik} as defined in (5.10). Letting $C \triangleq A + \delta_M \cdot B$, equation (5.14) can be solved for $\vec{L^*}$ provided that C^{-1} exists (cf. comment (1) below). Hence

$$\vec{L^*} = Q \cdot L^*(\lambda) \cdot \vec{u^*} + L^{*(1)}(\lambda) \cdot \vec{v^*} + \vec{w^*} \quad , \tag{5.16}$$

with $\quad \vec{u^*} = \vec{f^*} \cdot C^{-1} \quad , \quad \vec{v^*} = \vec{g^*} \cdot C^{-1} \quad$ and $\quad \vec{w^*} = \vec{h^*} \cdot C^{-1} \quad ,$ (5.17)

where $\vec{u^*}$, $\vec{v^*}$ and $\vec{w^*}$ are row vectors with components u_k^*, v_k^*

and w_k^*, respectively. Note that like in Section IV $L^*(\lambda)$ and $L^{*(1)}(\lambda)$ are two unknown constants. To determine these, we substitute (5.16) together with (3.23) in definition (3.18) whence we obtain

$$L^*(z) = 1 + Q\cdot L^*(\lambda)\cdot u^*(z) + L^{*(1)}(\lambda)\cdot v^*(z) + w^*(z) \quad , \tag{5.18}$$

and, upon differentiating once wrt z,

$$L^{*(1)}(z) = Q\cdot L^*(\lambda)\cdot u^{*(1)}(z) + L^{*(1)}(\lambda)\cdot v^{*(1)}(z) + w^{*(1)}(z) \quad , \tag{5.19}$$

with

$$u^*(z) \triangleq \sum_{k=2}^{\infty} u_k^*\cdot z^k \quad , \quad u^{*(1)}(z) = \sum_{k=2}^{\infty} k\cdot u_k^*\cdot z^{k-1} \quad , \tag{5.20}$$

and $v^*(z)$, $v^{*(1)}(z)$, $w^*(z)$ and $w^{*(1)}(z)$ defined analogous to $u^*(z)$ and $u^{*(1)}(z)$. Evaluating (5.19) at $z=\lambda$ and solving for $L^{*(1)}(\lambda)$ yields

$$L^{*(1)}(\lambda) = \frac{Q\cdot L^*(\lambda)\cdot u^{*(1)}(\lambda) + w^{*(1)}(\lambda)}{1-v^{*(1)}(\lambda)} \quad . \tag{5.21}$$

Substituting this last equation in (5.18) we end up with

$$L^*(z) = 1 + Q\cdot L^*(\lambda)\cdot d^*(z) + d_w^*(z) \quad , \tag{5.22}$$

where $\quad d^*(z) \triangleq u^*(z) + K_L(\lambda)\cdot u^{*(1)}(\lambda)\cdot v^*(z) \quad ,$ (5.23)

$\qquad\quad d_w^*(z) \triangleq w^*(z) + K_L(\lambda)\cdot w^{*(1)}(\lambda)\cdot v^*(z) \quad ,$ (5.24)

and $\quad K_L(\lambda) \triangleq 1/(1-v^{*(1)}(\lambda)) \quad .$ (5.25)

Letting $z=\lambda$ in (5.22) we can solve for our second unknown quantity and we get

$$L^*(\lambda) = \frac{1+d_w^*(\lambda)}{1-Q\cdot d^*(\lambda)} \quad . \tag{5.26}$$

Again, the quantity which determines λ_{crit} is the denominator which appears in (5.26) and we define λ_{crit} as the smallest positive λ which satisfies

$$d^*(\lambda) = Q^{-1} \quad , \qquad Q > 1 \quad , \tag{5.27}$$

where $d^*(\lambda)$ is obtained from (5.23). Finally, we obtain an equation for L_N by equating coefficients of z^k on both sides of (5.22) and by substituting the resulting L_k^* in (3.19), i.e.,

$$L_N = 1 + Q \cdot L^*(\lambda) \cdot \sum_{k=2}^{N} \frac{N!}{(N-k)!} \cdot d_k^* + \sum_{k=2}^{N} \frac{N!}{(N-k)!} \cdot d_{wk}^* \quad , \; N \geqslant 2 \; , \tag{5.28}$$

$$\text{with} \quad d_k^* \triangleq u_k^* + K_L(\lambda) \cdot u^{*(1)}(\lambda) \cdot v_k^* \quad , \quad k \geqslant 2 \quad , \tag{5.29}$$

$$d_{wk}^* \triangleq w_k^* + K_L(\lambda) \cdot w^{*(1)}(\lambda) \cdot v_k^* \quad , \quad k \geqslant 2 \quad , \tag{5.30}$$

and initial values (3.14). Note that $S^*(\lambda)$ and S_N (and, in principle, the corresponding expressions for all higher moments) can be computed by following the same steps as we did above for $L^*(\lambda)$ and L_N, see Appendix B for the resulting expressions.

Comments:

(1) To obtain equations (5.16) and (B.23) one needs the inverse of the doubly semi-infinite matrix C. A thorough discussion of the conditions under which C^{-1} exists turns out to be rather complex. To compute C^{-1} numerically, we inverted the truncated version of C and checked whether C^{-1} converged elementwise as we increased the truncation index R in the range R=10..

..40. For values of $p_j \leqslant .9$, all $j \in \{1,2,...,Q\}$, $\lambda \leqslant .6$ and $Q \geqslant 2$ we observed excellent convergence for $R \approx 16..30$; most of the practically interesting (i.e., $\lambda \leqslant \lambda_{crit}$) results can be computed to a precision of at least 6 decimal digits with R=16. It is worth noting that the inversion of matrix C is not affected (in the sense of matrix C being close to singular) if λ is close to or even above λ_{crit}.

(2) From equation (5.26) we deduce that $L^*(\lambda)$ and therefore (by (5.28)) the L_N exist if and only if $\lambda < \lambda_{crit}$ and $Q>1$ (the second condition follows trivially from the fact that a group of colliding transmitters cannot be split up into smaller groups if $Q \leqslant 1$). Similarly, it can be shown that under the above conditions all higher moments of Y_N exist, see Appendix B for the second moment. As was the case in the previous section, the only solution for the L_N if $\lambda \geqslant \lambda_{crit}$ is $L_N = \infty$ for $N \geqslant 2$.

(3) The quantities $L^*(\lambda)$ and $S^*(\lambda)$ (cf. Appendix B) of course admit the same interpretation as the unconditional first and second moment of the CRI-length as we pointed it out in comment (3), Section IV.

(4) Biased coins. In the light of the asymmetry of equation (5.4) it does not come as a big surprise that the modified free access RAS's perform somewhat bet-

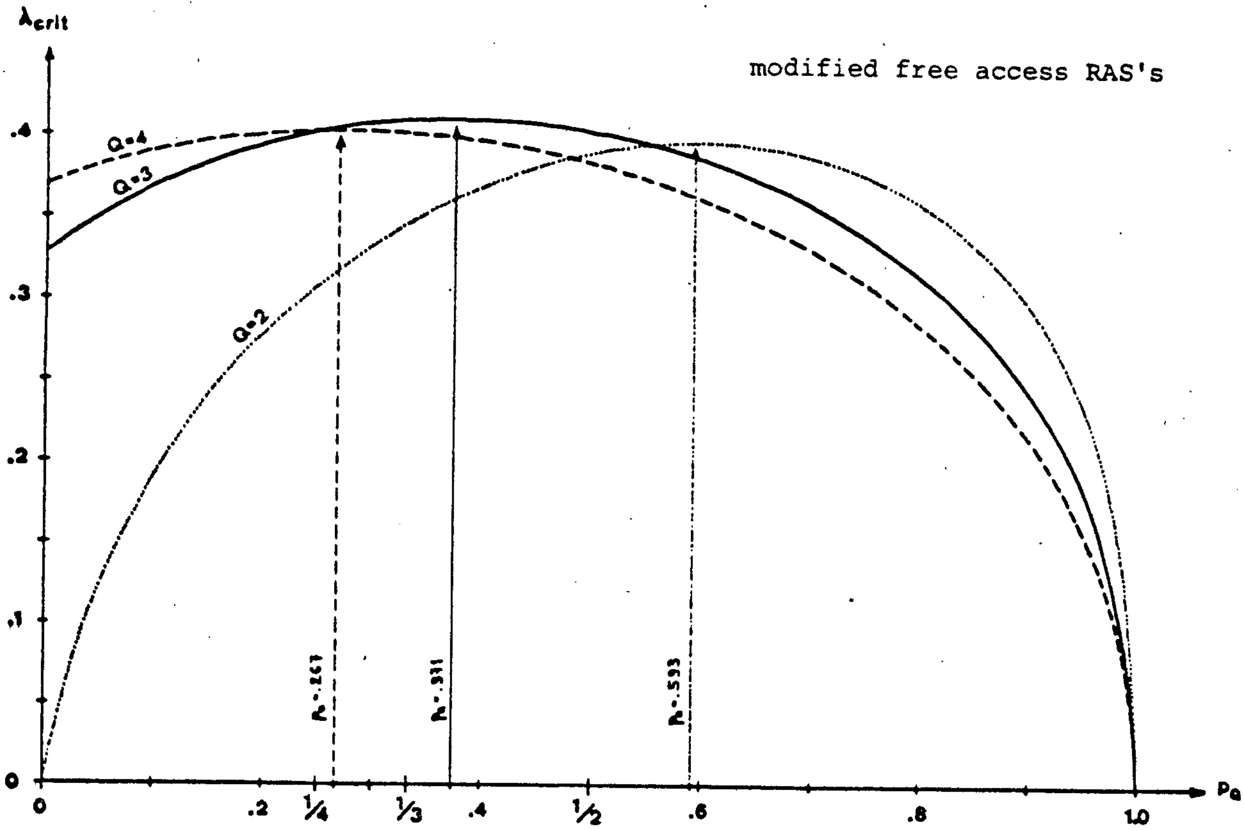

Fig. 5.2 Biased coins: λ_{crit} as a function of p_a, the probability that the value Q is flipped. All other values (1..Q-1) appear with probability p, where $p=(1-p_a)/(Q-1)$.

ter when biased coins with $p_a > Q^{-1}$ are utilized. We plotted λ_{crit} versus p_a in Fig. 5.2; for numerical values, see table 5.2.

Iteration method. There seems to be no principal reason which prevents the applicability of the iteration method for the modified free access RAS's. For the sake of relative simplicity we give some clues for the solution of (5.4) when fair coins are used. Let $\sigma_i(z)=\lambda_i+p\cdot z$, $i\in\{1,2,3\}$, be distinct (but not necessarily different) Moebius transforms (cf. Appendix A). Then, under certain contraction conditions, the general equation (where d(z) is an entire function)

$$t(z) - \sum_{i=1}^{3} a_i \cdot e^{-\alpha_i \cdot \sigma_i(z)} \cdot t(\sigma_i(z)) = d(z) \quad , \qquad (5.31)$$

has the unique, entire solution

$$t(z) = d(z) + \sum_{\ell=1}^{\infty} \sum_{i_1=1}^{3} \cdots \sum_{i_\ell=1}^{3} a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot a_{i_\ell} \times$$

$$\times \hat{d}(\sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_\ell}(z)) \quad , \qquad (5.32)$$

where the ^ operator is defined as

$$\hat{\Psi}(\sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_\ell}(z)) \triangleq e^{-\alpha_{i_1} \cdot \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_\ell}(z)} \cdot e^{-\alpha_{i_2} \cdot \sigma_{i_2} \ldots \sigma_{i_\ell}(z)} \times$$

$$\times \ldots \cdot e^{-\alpha_{i_\ell} \cdot \sigma_{i_\ell}(z)} \cdot \Psi(\sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_\ell}(z)) \; ,$$

(5.33)

and $$\sigma_{i_j} \sigma_{i_k}(z) \triangleq (\sigma_{i_j} \circ \sigma_{i_k})(z) = \sigma_{i_j}(\sigma_{i_k}(z)) \quad . \qquad (5.34)$$

The set of all substitutions $\sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_\ell}(z)$, $\ell = 0, 1, 2, \ldots$, (where ℓ is called the substitution length) form a non-commutative semigroup under the operation of composition of functions (cf. Fig. 5.3) with identity element ($\ell = 0$)

$$\varepsilon(z) = z \quad . \qquad (5.35)$$

Note that

$$\sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_\ell}(z) = \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_\ell}(0) + p^{\ell} \cdot z \quad , \qquad (5.36)$$

and

$$\hat{\Psi}^{(1)}(\sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_\ell}(z)) = e^{-\alpha_{i_1} \cdot \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_\ell}(z)} \cdot e^{-\alpha_{i_2} \cdot \sigma_{i_2} \ldots \sigma_{i_\ell}(z)} \times$$

$$\times \ldots \cdot e^{-\alpha_{i_\ell} \cdot \sigma_{i_\ell}(z)} \cdot [\Psi^{(1)}(\sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_\ell}(z)) +$$

$$- (\alpha_{i_1} + \frac{\alpha_{i_2}}{p} + \ldots + \frac{\alpha_{i_\ell}}{p^{\ell-1}}) \cdot \Psi(\sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_\ell}(z))] \; . \qquad (5.37)$$

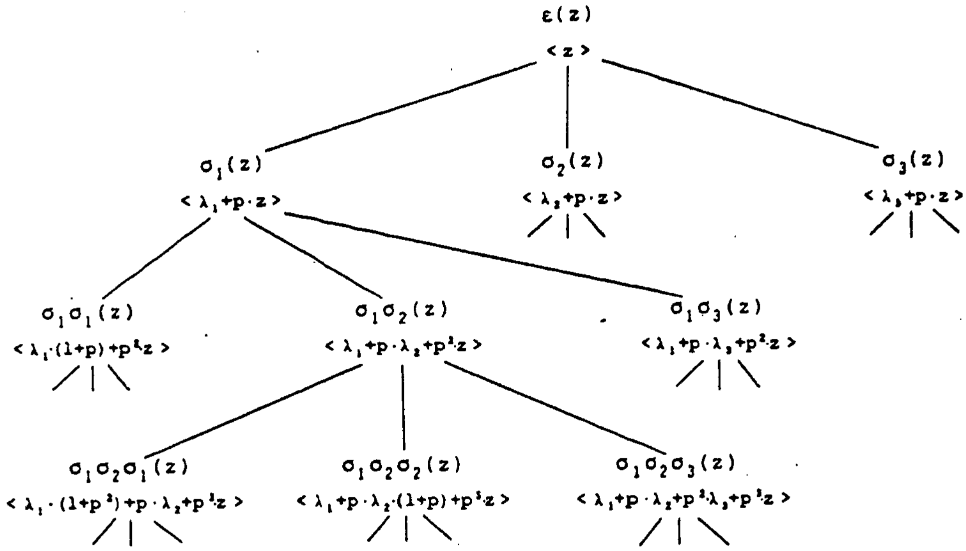

Fig. 5.3 The non-commutative semigroup which is generated by σ_1, σ_2 and σ_3 under the operation of composition of functions.

In the case of the modified free access RAS's with fair coins the parameters in equation (5.31) are

$$a_1 = Q \quad , \quad \alpha_1 = 0 \quad , \quad \sigma_1(z) = \lambda + Q^{-1}\cdot z \quad , \qquad (5.38)$$

$$a_2 = -\delta_M \quad , \quad \alpha_2 = Q-1 \quad , \quad \sigma_2(z) = \lambda + Q^{-1}\cdot z \quad , \qquad (5.39)$$

$$a_3 = \delta_M \cdot P_0 \quad , \quad \alpha_3 = Q-1 \quad , \quad \sigma_3(z) = Q^{-1}\cdot z \quad , \qquad (5.40)$$

and, if we restrict ourselves to the first moment of the CRI-length,

$$t(z) = L^*(z) \quad , \qquad (5.41)$$

$$d(z) = Q\cdot L^*(\lambda)\cdot f^*(z) + L^{*(1)}(\lambda)\cdot g^*(z) + h^*(z) \quad , \qquad (5.42)$$

with $f^*(z)$, $g^*(z)$ and $h^*(z)$ as defined in (5.5)...(5.7) (with p_a replaced by Q^{-1}). If we iterate the basic equation (5.31) m times, substitute (5.41) and take care of the initial conditions (3.23) we certainly have

$$L^*(z) - 1 = \hat{\mathfrak{R}}_{m-1}(d(.);z) + \hat{\mathfrak{S}}_m(L^*(.);z) \quad ,$$

$$m = 1,2,\ldots, \qquad (5.43)$$

where the operators $\hat{\mathfrak{R}}_m$ and $\hat{\mathfrak{S}}_m$ are defined in the following way

$$\hat{\mathfrak{S}}_m(\Psi(.);z) \triangleq \sum_{i_1=1}^{3} \cdots \sum_{i_m=1}^{3} a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot a_{i_m} \cdot [\hat{\Psi}(\sigma_{i_1}\sigma_{i_2} \ldots \sigma_{i_m}(z)) +$$
$$- \hat{\Psi}(\sigma_{i_1}\sigma_{i_2} \ldots \sigma_{i_m}(0)) - z \cdot Q^{-m} \cdot \hat{\Psi}^{(1)}(\sigma_{i_1}\sigma_{i_2} \ldots \sigma_{i_m}(0))] \,,$$
$$(5.44)$$

$$\hat{\mathfrak{S}}_0(\Psi(.);z) \triangleq \Psi(z) - \Psi(0) - z \cdot \Psi^{(1)}(0) \quad , \qquad (5.45)$$

$$\hat{\mathfrak{R}}_m(\Psi(.);z) \triangleq \sum_{\ell=0}^{m} \hat{\mathfrak{S}}_\ell(\Psi(.);z) \quad . \qquad (5.46)$$

Note the analogy between the $\mathfrak{R}$ operator which we defined in (3.36) and $\hat{\mathfrak{R}}_m$ as defined above as $m \to \infty$. Equation (5.43) suggests the following solution for equation (5.4) when fair coins are utilized

$$L^*(z) = 1 + Q \cdot L^*(\lambda) \cdot \hat{\mathfrak{R}}_\infty(f^*(.);z) + L^{*(1)}(\lambda) \cdot \hat{\mathfrak{R}}_\infty(g^*(.);z) +$$
$$+ \hat{\mathfrak{R}}_\infty(h^*(.);z) \quad , \qquad (5.47)$$

provided that

$$\lim_{m \to \infty} \hat{\mathfrak{S}}_m(L^*(.);z) = 0 \quad , \qquad (5.48)$$

and $$\lim_{m \to \infty} \hat{\mathfrak{R}}_m(d(.);z) < \infty \quad . \qquad (5.49)$$

It should be emphasized that equation (5.47) is a conjecture since we did not prove (5.48) and (5.49) so far. To do so, it is not enough to rely solely on some form of contraction condition (like we did in Appendix A); the special form of the term preceded by δ_M in equation (5.4) suggests, however, the additional specification of an "annihilation condition", i.e., we can expect that most of the iterates of some large enough substitution length ℓ will cancel out due to the minus sign of the factor a_2 in (5.39). To complete the solution of (5.4), one must of course determine the constants $L^*(\lambda)$ and $L^{*(1)}(\lambda)$ in (5.47). This can be done by the same method which we used to obtain (4.15) and (4.22).

Numerical results. Here we give again the numerical values of λ_{crit} for Q=2..10. In addition, we computed λ_{crit} and p_a for optimally biased coins for Q=2..4. Numerical values for the other quantities which we analyzed in this section can be found in Appendix C. Computations were again done

Q	λ_{crit}
2	.387222
3	.406970
4	.400746
5	.387780
6	.373576
7	.359853
8	.347053
9	.335330
10	.324619

Table 5.1 λ_{crit} as a function of Q for the modified free access RAS's with fair coins (the systems are stable for all $\lambda < \lambda_{crit}$).

by the same means as depicted in Section III. Numerical stability (with the direct method) is satisfactory for

Q	optimum p_Q	λ_{crit}
2	.593200	.393225
3	.370911	.407614
4	.266662	.400851

Table 5.2 λ_{crit} and p_Q for the modified free access RAS's with optimally biased coins. The coin values 1,2,...,Q-1 are equally likely and have probability p, with $p=(1-p_Q)/(Q-1)$.

moderate N (6 to 7 digits of the results are precise up to N≈25 when Fortran double precision is used) with a truncation point R≈20..30.

Comments:

(5) The modified free access RAS's display the same sensitivity on the mean arrival rate λ as described in Section IV, comment (7) for the basic free access RAS's. Thus, the indication of λ is essential for the interpretation of numerical results.

(6) As might be expected, the modified free access RAS's outrival the other three RAS's which we considered in this paper (cf. Section VI). Again, Q=3 is the best choice, although, this time the superiority of Q=3 over Q=2 is less pronounced than it was for the basic free access RAS's. For $\lambda>.3$, the conditional expected CRI-length is minimized for all $N\geq 2$ if Q=3. Note that if one plots $L^*(\lambda)$ versus Q for various rates λ then one obtains a figure similar to Fig. 4.3.

VI. Conclusions

We have analyzed the throughput characterisitcs of four different classes of RAS's whose salient features are their simplicity (in terms of implementation) and, as opposed to the well known ALOHA system, their inherent stability (under the assumption of a Poisson arrival process) for arrival rates λ less than λ_{crit}. At this point, a comparison of the four classes of RAS's intrudes itself. We made allowance for this by including Fig. 6.1 which de-

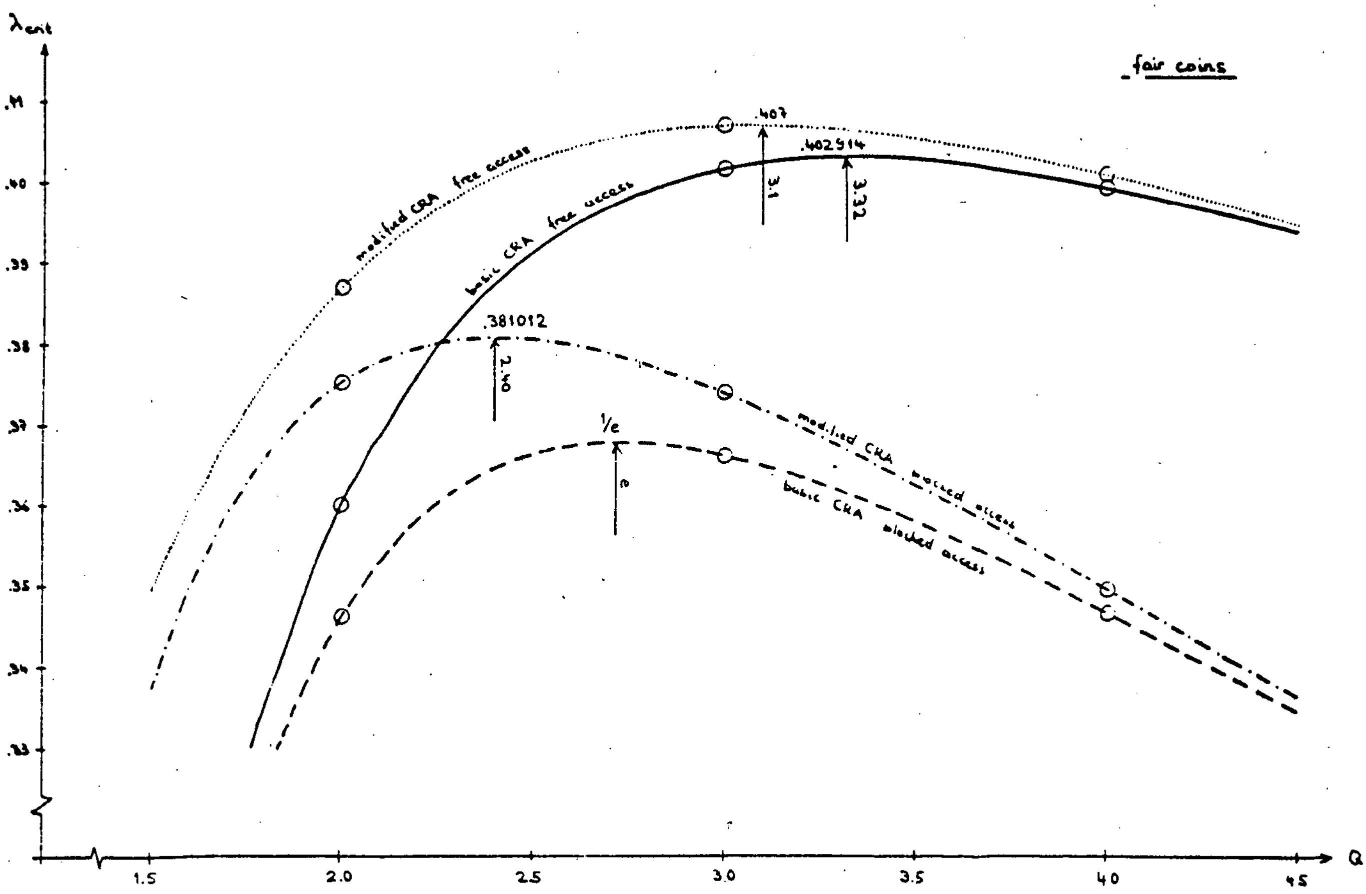

Fig. 6.1 λ_{crit} versus Q for the four RAS's which we analyzed in this paper.

picts λ_{crit} versus the splitting parameter Q (in the range of interest) for the basic and the modified CRA combined with either the free or the blocked access CAP. What immediately catches the eye is that for Q say greater than 3 it is hardly worthwile to worry about the modified CRA, the important consideration here is whether to use the blocked or the free access CAP. For Q between 1 and 2 (if this could be realized somehow), on the other hand, the dominant factor is the CRA and here it certainly pays to apply the modified CRA, although, as we already stressed earlier, this requires a ternary feedback channel instead of only a binary one. This supports our intuition which tells us that for large Q a considerable amount of empty slots is created which can be exploited by the new arrivals under the free access CAP whereas for small Q the collision slots prevail such that the modification of the CRA can take effect. All told, the favourite RAS which excels by its simplicity while offering good throughput characteristics (λ_{crit}=.4016) is the basic free access RAS with Q=3.

Besides the more system oriented quantity λ_{crit} any RAS must also be judged by its more user oriented behaviour; a good measure for this is the first and second moment of the packet delay. We did not touch upon this issue in the current paper, mainly for reasons of space. For the basic free access RAS's with Q=2 the delay question has been addressed in [FAY-FLA-HOF-JAC83] and, independently, in [VVE-TSY84]; for Q>2 we refer to the forthcoming paper

[MAT-TSY-VVE84]. To give a flavour of the delay characteristics and to further justify our choice of the basic free access RAS with Q=3 we plotted the mean and the standard deviation of the packet delay versus λ for RAS's which use the basic CRA with Q=2 and Q=3 (fair coins) combined with either the obvious BAP or the FAP, see Fig. 6.2 and Fig. 6.3. We obtained the two figures by simulating 50'000 slots

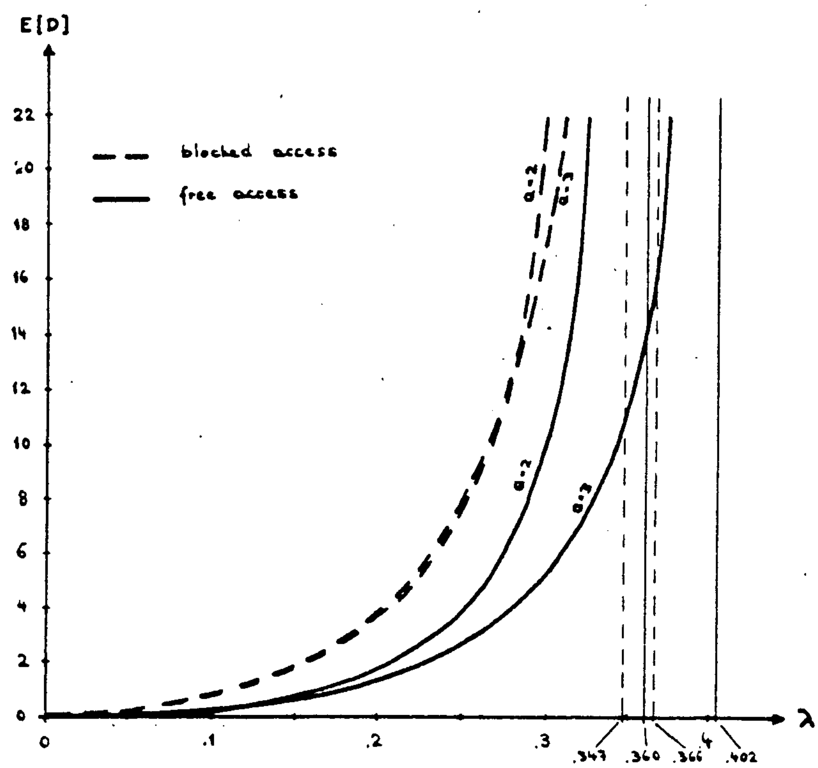

Fig. 6.2 Mean packet delay E[D] for various rates λ when the basic CRA with fair coins is combined with either the blocked or the free access CAP. The results were obtained by simulating 50'000 slots.

for each value of λ and each of the four RAS's; our thanks for carrying out the simulations are due to B.V. Faltings. The most striking feature (with respect to the packet delay) of the basic free access RAS with Q=3 is that it not

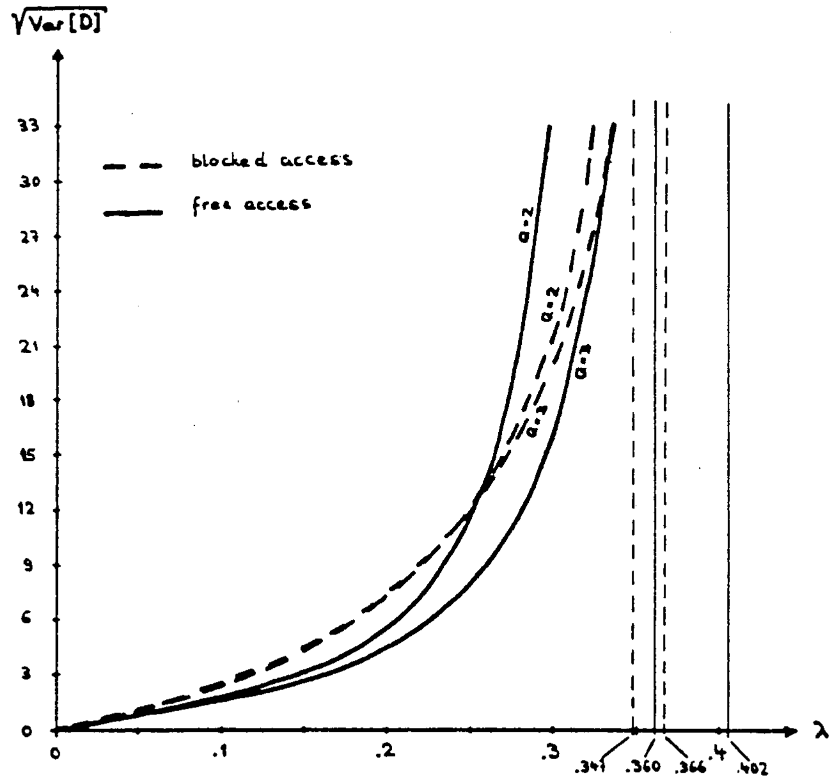

Fig. 6.3 Standard deviation of the packet delay D for various rates λ when the basic CRA with fair coins is combined with either the blocked or the free access CAP. The results were obtained by simulating 50'000 slots.

only has good first order properties but also second order properties which outperform those which can be achieved by selecting Q=2.

As to the mathematical tools which we provided in this paper, we believe that they should be applicable (some in fact have been applied) to a number of problems which are recursive in nature. For analytical purposes, the iteration method is clearly head and shoulders above the direct method; if one is merely interested in numerical results for certain quantities, like $L^*(\lambda)$ or $S^*(\lambda)$ in our case, then the direct method can be as precise but more efficient than the iteration method (in our context this holds before all for the free access RAS's when biased coins are used).

Finally, we would like to point out that one must be cautious in applying the results of the free access RAS's (with the Poisson arrival assumption !) to systems with a finite number of transmitters (if packets must be sent sequentially), see [FAL83]. Just to give an example, assume a system with only three transmitters. It is then possible, that the total load offered to the common channel is mainly made up by packets transmitted from two of the three stations. In such a case it can happen that the third transmitter, once he has to put one of his packets onto the stack, sinks infinitely deep into the stack although the total arrival rate λ does not exceed λ_{crit}. This comes from the fact that in any system with only a finite number of (active) transmitters, the state can be reached where there are just simply no more (active) transmitters with newly arrived packets to take advantage of empty slots. Hence, λ_{crit} drops to the value of the corresponding blocked access RAS. Instability of the channel and unfairness of the system can be prevented in the finite transmitter case by the introduction of the free access, blocked exit CAP which essentially states the following (in addition to the free access CAP and the CRA). Every transmitter, if he is involved in a collision, has to simulate the behaviour of the "worst case transmitter" (i.e., the transmitter who always flips the value Q on his coin) who is involved in the same collision until this unlucky fellow had a chance to transmit his packet. It is only after the occurence of this event that the above cited transmitters are allowed to reaccept new packets for the first-time transmission; for a more detailed description, see [MAT84]. It remains to note that also for the free access, blocked exit RAS's the choice Q=3 is optimum.

Acknowledgement

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Appendix A

Iteration Scheme to Solve the Functional Equations

Here we develop an iterative scheme to solve the basic funtional equation[+)]

$$t(z) - \gamma \cdot t(\lambda + p \cdot z) = d(z) \quad , \tag{A.1}$$

where λ and p are real constants, with $0<|p|<1$. Let

$$M = \begin{bmatrix} p & \lambda \\ 0 & 1 \end{bmatrix} \tag{A.2}$$

be a Moebius transformation consisting of a rotation p (applied first) and a translation λ [HEN74]. With M we associate the function

$$\sigma_M(z) = \lambda + p \cdot z \quad . \tag{A.3}$$

Substituting z in (A.3) m times with $\sigma_M(z)$ yields the m-th iterate

$$\sigma_M^{[m]}(z) = \lambda \cdot \frac{1 - p^m}{1 - p} + p^m \cdot z = \sigma_M^{[m]}(0) + p^m \cdot z \; , \tag{A.4}$$

with substitution length m. The set of all substitutions

+) We use the term "functional equation" to denote an equation between two expressions which are built from a finite number of functions (of which at least one is unknown) and variables by a finite number of superpositions. A simple example of a functional equation is $\Psi(1+z)=z\cdot\Psi(z)$ which (under appropriate conditions) admits the gamma function as its solution.

$\sigma_H^{[m]}$ for $m \geq 0$ form a commutative semigroup H under the operation of composition of functions, i.e.,

$$(\sigma_H^{[m_2]} \circ \sigma_H^{[m_1]})(z) = \sigma_H^{[m_2]}[\sigma_H^{[m_1]}(z)] = \sigma_H^{[m_1+m_2]}(z) \quad . \tag{A.5}$$

The identity element of H(z) is

$$\sigma_H^{[0]}(z) = z \quad , \tag{A.6}$$

whereas the accumulation point of H is

$$\sigma_H^{[\infty]}(\cdot) = \frac{\lambda}{1-p} \quad . \tag{A.7}$$

For convenience we shall use the notations

$$\lambda_m \triangleq \sigma_H^{[m]}(0) = \lambda \cdot \frac{1-p^m}{1-p} \quad , \tag{A.8}$$

and
$$\mu \triangleq \sigma_H^{[\infty]}(\cdot) = \frac{\lambda}{1-p} \quad . \tag{A.9}$$

With these prerequisites in mind we can now state: If γ in (A.1) satisfies the contraction condition

$$|\gamma| < 1 \tag{A.10}$$

and if d(z) in (A.1) is an entire function then the functional equation (A.1) has a unique, entire solution given by

$$t(z) = \sum_{m=0}^{\infty} \gamma^m \cdot d[\sigma_H^{[m]}(z)] \quad . \tag{A.11}$$

This is easily verified by letting

$$\gamma \cdot t(\lambda + p \cdot z) = \gamma \cdot \sum_{m=0}^{\infty} \gamma^m \cdot d[\sigma_H^{[m]}(\lambda + p \cdot z)] = \sum_{m=0}^{\infty} \gamma^{m+1} \cdot d[\sigma_H^{[m+1]}(z)] \tag{A.12}$$

and therefore

$$t(z) - \gamma\cdot t(\lambda+p\cdot z) = d[\sigma_M^{[0]}(z)] = d(z) \quad . \tag{A.13}$$

For a proof of the existence and unicity of (A.11), see [FAY-FLA-HOF82]. It should be noted here that the iteration scheme described above extends to the use of several, different Moebius transformations to solve equations of the type

$$t(z) - \sum_{j=1}^{Q} \gamma_j\cdot t(\lambda+p\cdot z) = d(z) \quad . \tag{A.14}$$

The resulting expressions, however, are not easy to treat in general - for a case with Q=2, see [FAY-FLA-HOF82], for Q$\geqslant$2, see [MAT84]. For a general reference on functional equations in one variable we refer to [KUC68].

Appendix B

Expressions for the second moment of Y_N

Here we list (without derivation) some of the expressions for the conditional second moment of the CRI-length.

Blocked access RAS's

$f_L^*(z)$ in equation (3.27) is

$$f_L^*(z) = 2\cdot\{L^*(z)-1+\sum_{j=1}^{Q-1}\sum_{h=j+1}^{Q} L^*(p_j\cdot z)\cdot L^*(p_h\cdot z)\}-e^{-z}\cdot Q\cdot(Q-1)\cdot(1+z)+$$
$$-\ \delta_M\cdot 2\cdot\{e^{-z\cdot(1-p_Q)}\cdot[L^*(p_Q\cdot z)+Q-2] - e^{-z}\cdot(Q-1)\cdot(1+p_Q\cdot z)\}\ . \qquad \text{(B.1)}\ \square$$

Direct method: S_k^* and S_N (cf. (3.29)...(3.31)).

$$S_k^* = \frac{Q\cdot f_k^* + h_k^* + f_{Lk}^*}{1-\sum_{j=1}^{Q} p_j^{\,k}}\ , \quad k \geq 2\ , \qquad \text{(B.2)}$$

with $S_0^* = 1$, $S_1^* = 0$, (B.3)

and f_k^*, h_k^* and f_{Lk}^* defined as the coefficients of the power series representations of $f^*(z)$, $h^*(z)$ (eqs. (3.21), (3.22)) and $f_L^*(z)$ (eq. (B.1)), respectively. Using (3.26) one obtains

$$S_N = 1 + \sum_{k=2}^{N}\frac{2\cdot N!}{(N-k)!\cdot(1-\sum_{j=1}^{Q} p_j^{\,k})}\cdot\Big[\sum_{i=0}^{k} L_i^*\cdot L_{k-i}^*\cdot\sum_{j=1}^{Q-1}\sum_{h=j+1}^{Q} p_j^{\,i}\cdot p_h^{\,k-i}\ +$$
$$+\ L_k^* - \delta_M\cdot\sum_{i=2}^{k}\frac{(p_Q-1)^{k-i}}{(k-i)!}\cdot p_Q^{\,i}\cdot L_i^*\Big]\ +$$
$$+\sum_{k=2}^{N}\binom{N}{k}\cdot\frac{(-1)^k\cdot\{Q^2\cdot(k-1)-\delta_M\cdot(2\cdot Q-1)\cdot[k\cdot p_Q+(1-p_Q)^k-1]\}}{1-\sum_{j=1}^{Q} p_j^{\,k}}\ ,$$

$$N \geqslant 2 \quad , \tag{B.4}$$

with initial values $S_0 = S_1 = 1$. (B.5) □

<u>Iteration method: S_N for fair coins</u> (cf. (3.40)).

$$S_N = 1 + (Q-1)\cdot\sum_{m=1}^{\infty} Q^m \cdot \sum_{\substack{k,\ell,u \\ k+\ell\geq 2}}^{N} \binom{N}{k\ \ell\ u}\cdot L_k\cdot L_\ell\cdot (Q^{-m})^{k+\ell}\cdot(1-2\cdot Q^{-m})^u +$$
$$+ 2\cdot\sum_{m=0}^{\infty} Q^m\cdot\sum_{k=2}^{N}\binom{N}{k}\cdot L_k\cdot(Q^{-m})^k\cdot(1-\delta_M\cdot Q^{-k})\cdot(1-Q^{-m})^{N-k} +$$
$$- (Q-1)\cdot\sum_{m=1}^{\infty} Q^m\cdot[1-(1-2\cdot Q^{-m})^N - 2\cdot N\cdot Q^{-m}\cdot(1-2\cdot Q^{-m})^{N-1}] +$$
$$+ (Q^2-2)\cdot\sum_{m=0}^{\infty} Q^m\cdot[1-(1-Q^{-m})^N - N\cdot Q^{-m}\cdot(1-Q^{-m})^{N-1}] +$$
$$- \delta_M\cdot(2\cdot Q-3)\cdot\sum_{m=0}^{\infty} Q^m\cdot\{[1-Q^{-m}\cdot(1-Q^{-1})]^N - (1-Q^{-m})^N +$$
$$-N\cdot Q^{-m-1}\cdot(1-Q^{-m})^{N-1}\} \quad , \quad N \geqslant 2 \; . \tag{B.6}$$
□

Basic free access RAS's

<u>Tgf for the second moment of Y_N</u> (arbitrarily biased coins).

$$S^*(z) - \sum_{j=1}^{Q} S^*(\lambda+p_j\cdot z) = 1+Q\cdot S^*(\lambda)\cdot f^*(z)+S^{*(1)}(\lambda)\cdot g^*(z)+\rho_L^*(z) \tag{B.7}$$

with initial conditions (3.28), $f^*(z)$ and $g^*(z)$ as defined in (4.8) and (4.9) and

$$\rho_L^*(z) = 2\cdot[L^*(z) - 1 + \sum_{j=1}^{Q-1}\sum_{h=j+1}^{Q} L^*(\lambda+p_j\cdot z)\cdot L^*(\lambda+p_h\cdot z)] +$$
$$- (Q-1)\cdot L^*(\lambda)\cdot e^{-z}\cdot[Q\cdot L^*(\lambda)\cdot(1+z) + 2\cdot L^{*(1)}(\lambda)\cdot z] \tag{B.8}$$
□

Iteration method: Solution of (B.7) for fair coins.

$$S^*(z) = 1 + Q\cdot S^*(\lambda)\cdot d^*(z) + K_S(\lambda)\cdot \Re(g^*(.);z) + \Re(f_L^*(.);z) , \tag{B.9}$$

with $d^*(z)$ as defined in (4.19),

$$S^*(\lambda) = L^*(\lambda)\cdot[1+K_S(\lambda)\cdot\Re(g^*(.);\lambda)+\Re(f_L^*(.);\lambda)] , \tag{B.10}$$

$$K_S(\lambda) \triangleq Q\cdot L^*(\lambda)\cdot K_L(\lambda)\cdot[2\cdot L^{*(1)}(\mu)\cdot K_L(\lambda)-\mu\cdot(Q-1)\cdot L^*(\lambda)] , \tag{B.11}$$

$$L^{*(1)}(\mu) = Q\cdot L^*(\lambda)\cdot K_L(\lambda)\cdot \mu\cdot e^{-\mu}\cdot\sum_{m=0}^{\infty} Q^{-m}\cdot e^{\mu\cdot Q^{-m}} , \tag{B.12}$$

$$\Re(g^*(.);\lambda) = \sum_{m=0}^{\infty} Q^m\cdot[(1-\lambda_m)\cdot\lambda\cdot Q^{-m}\cdot e^{-\lambda_m}+\lambda_m\cdot e^{-\lambda_m}-\lambda_{m+1}\cdot e^{-\lambda_{m+1}}] , \tag{B.13}$$

$$\lambda_m \triangleq \sigma_N^{[m]}(0) = \mu\cdot(1-Q^{-m}) , \tag{B.14}$$

and

$$\Re(f_L^*(.);\lambda) = 2\cdot\sum_{m=0}^{\infty} Q^m\cdot[L^*(\lambda_{m+1})-L^*(\lambda_m)-Q^{-m}\cdot\lambda\cdot L^{*(1)}(\lambda_m)] +$$
$$+(Q-1)\cdot\sum_{m=1}^{\infty} Q^m\cdot\{[L^*(\lambda_{m+1})]^2-[L^*(\lambda_m)]^2-2\cdot Q^{-m}\cdot\lambda\cdot L^*(\lambda_m)\cdot L^{*(1)}(\lambda_m)\}+$$
$$+ Q\cdot(Q-1)\cdot[L^*(\lambda)]^2\cdot[d^*(\lambda)+\mu\cdot K_L(\lambda)\cdot\Re(g^*(.);\lambda)] . \tag{B.15}$$

For the quantities μ, $L^*(\lambda)$, $K_L(\lambda)$ and $d^*(\lambda)$, see (4.12), (4.22), (4.20) and (4.23), respectively. □

Iteration method: S_N for fair coins (cf. (4.25)).

$$S_N = 1 + (Q-1)\cdot\{\sum_{m=1}^{\infty} e^{-2\cdot\lambda_m}\cdot Q^m\cdot\sum_{\substack{k,\ell,u\\k+\ell\geq 2}}^{N} \binom{N}{k\ \ell\ u}\cdot L^{(k)}(\lambda_m)\cdot L^{(\ell)}(\lambda_m)\times$$
$$\times(Q^{-m})^{k+\ell}\cdot(1-2\cdot Q^{-m})^u +$$
$$-\sum_{m=1}^{\infty}[L(\lambda_m)]^2\cdot e^{-2\lambda_m}\cdot Q^m\cdot[1-2\cdot N\cdot Q^{-m}-(1-2\cdot Q^{-m})^N] +$$
$$-2\cdot N\cdot\sum_{m=1}^{\infty} L(\lambda_m)\cdot L^{(1)}(\lambda_m)\cdot e^{-2\cdot\lambda_m}\cdot[1-(1-2\cdot Q^{-m})^{N-1}]\} +$$

$$+ \; 2\cdot\{\sum_{m=0}^{\infty} e^{-\lambda_m}\cdot Q^m \cdot \sum_{k=2}^{N} \binom{N}{k}\cdot L^{(k)}(\lambda_m)\cdot(Q^{-m})^k\cdot(1-Q^{-m})^{N-k} \; +$$

$$- \sum_{m=0}^{\infty} L(\lambda_m)\cdot e^{-\lambda_m}\cdot Q^m\cdot[1-N\cdot Q^{-m}-(1-Q^{-m})^N] \; +$$

$$- \; N\cdot\sum_{m=0}^{\infty} L^{(1)}(\lambda_m)\cdot e^{-\lambda_m}\cdot[1-(1-Q^{-m})^{N-1}]\} \; +$$

$$+ \; Q\cdot\{(Q-1)\cdot[L^*(\lambda)]^2+S^*(\lambda)\}\cdot\sum_{m=0}^{\infty} e^{-\lambda_m}\cdot Q^m\cdot[1-N\cdot Q^{-m}-(1-Q^{-m})^N]+$$

$$+ \; \{Q\cdot(Q-1)\cdot[L^*(\lambda)]^2\cdot[2\cdot K_L(\lambda)-1]+Q\cdot S^*(\lambda)\cdot K_L(\lambda)+K_S(\lambda)\}\times$$

$$\times\{\sum_{m=0}^{\infty} \lambda_m\cdot e^{-\lambda_m}\cdot Q^m\cdot[1-N\cdot Q^{-m}-(1-Q^{-m})^N] \; +$$

$$+ \; N\cdot\sum_{m=0}^{\infty} e^{-\lambda_m}\cdot[1-(1-Q^{-m})^{N-1}]\} \quad ,$$

$$N \geqslant 2 \quad , \qquad (B.16)$$

with $L^*(\lambda)$, $S^*(\lambda)$, $K_L(\lambda)$, $K_S(\lambda)$ and λ_m as defined in (4.22), (B.10), (4.20), (B.11) and (B.14). For $L^{(k)}(\lambda_m)$, see the footnote of (3.23). □

Modified free access RAS's

Note: The following equations may be used for all four RAS's which we consider in this paper. For basic CRA's one simply has to set $\delta_M=0$; for the blocked access RAS's one has to set $\lambda=0$.

Tgf for the second moment of Y_N (arbitrarily biased coins).

$$S^*(z) - \sum_{j=1}^{Q} S^*(\lambda+p_j\cdot z) + \delta_M\cdot P_0\cdot e^{-z\cdot(1-p_Q)}\cdot[S^*(\lambda+p_Q\cdot z)-S^*(p_Q\cdot z)] =$$

$$= Q\cdot S^*(\lambda)\cdot f^*(z)+S^{*(1)}(\lambda)\cdot g^*(z)+h^*(z)+f_L^*(z) \quad , \qquad (B.17)$$

with initial conditions (3.28), $f^*(z)$, $g^*(z)$ and $h^*(z)$ as defined in (5.5)...(5.7) and

$$\rho_L^*(z) = 2\cdot[L^*(z) - 1 + \sum_{j=1}^{Q-1}\sum_{h=j+1}^{Q} L^*(\lambda+p_j\cdot z)\cdot L^*(\lambda+p_h\cdot z)] +$$

$$- (Q-1)\cdot L^*(\lambda)\cdot e^{-z}\cdot[Q\cdot L^*(\lambda)\cdot(1+z) + 2\cdot L^{*(1)}(\lambda)\cdot z] +$$

$$- \delta_M\cdot 2\cdot P_0\cdot\{e^{-z\cdot(1-p_Q)}\cdot[(Q-1)\cdot L^*(\lambda+p_Q\cdot z)-(Q-2)\cdot(L^*(p_Q\cdot z)-1)] +$$

$$- e^{-z}\cdot(Q-1)\cdot[L^*(\lambda)\cdot(1+p_Q\cdot z) + L^{*(1)}(\lambda)\cdot p_Q\cdot z]\}$$

(B.18) □

<u>Direct method: Solution of (B.17)</u> for arbitrarily biased coins (cf. (5.8)...(5.26)).

$$S_k^*\cdot a_{kk} + \sum_{i=k+1}^{\infty} S_i^*\cdot a_{ik} + \delta_M\cdot\sum_{i=2}^{\infty} S_i^*\cdot b_{ik} =$$

$$= Q\cdot S^*(\lambda)\cdot f_k^* + S^{*(1)}(\lambda)\cdot g_k^* + h_k^* + \rho_{Lk}^* \quad , \quad k \geq 2 \quad , \quad \text{(B.19)}$$

with

$$\rho_{Lk}^* = 2\cdot[L_k^* + \sum_{i=0}^{k}\frac{L^{*(k-i)}(\lambda)}{(k-i)!}\cdot\frac{L^{*(i)}(\lambda)}{i!}\cdot\sum_{j=1}^{Q-1}\sum_{h=j+1}^{Q} p_j^{k-i}\cdot p_h^{i}] +$$

$$+ \frac{(-1)^k}{k!}\cdot(Q-1)\cdot L^*(\lambda)\cdot[Q\cdot L^*(\lambda)\cdot(k-1) + 2\cdot k\cdot L^{*(1)}(\lambda)] +$$

$$- \delta_M\cdot 2\cdot P_0\cdot\{\sum_{i=0}^{k}\frac{(p_Q-1)^{k-i}}{(k-i)!}\cdot p_Q^{i}\cdot[(Q-1)\cdot\frac{L^{*(i)}(\lambda)}{i!} - (Q-2)\cdot L_i^*] +$$

$$+ (Q-2)\cdot\frac{(p_Q-1)^k}{k!} + \frac{(-1)^k}{k!}\cdot(Q-1)\cdot[L^*(\lambda)\cdot(k\cdot p_Q - 1) +$$

$$+ L^{*(1)}(\lambda)\cdot k\cdot p_Q]\} \quad , \quad k \geq 2 \quad , \quad \text{(B.20)}$$

$$L^{*(i)}(\lambda) = \sum_{l=i}^{\infty}\frac{l!}{(l-i)!}\cdot L_l^*\cdot\lambda^{l-i} \quad , \quad \text{(B.21)}$$

and a_{ik}, b_{ik}, f_k^*, g_k^* and h_k^* as defined in (5.9)...(5.13).

Equation (B.19) can be written as

$$\vec{S^*}\cdot(A + \delta_M\cdot B) = Q\cdot S^*(\lambda)\cdot\vec{f^*} + S^{*(1)}(\lambda)\cdot\vec{g^*} + \vec{h^*} + \vec{\rho_L^*} \quad , \quad \text{(B.22)}$$

with $\vec{S^*}$ and $\vec{\rho_L^*}$ defined analogously to (5.15). With $C \triangleq A + \delta_M\cdot B$, equation (B.22) can be solved for $\vec{S^*}$ (provided C^{-1} exists)

$$\vec{S^*} = Q\cdot S^*(\lambda)\cdot\vec{u^*} + S^{*(1)}(\lambda)\cdot\vec{v^*} + \vec{w^*} + \vec{\psi_L^*} \quad , \qquad (B.23)$$

with $\vec{\psi_L^*} = \vec{\varphi_L^*}\cdot C^{-1}$, (B.24)

and $\vec{u^*}$, $\vec{v^*}$ and $\vec{w^*}$ as defined in (5.17). Let

$$\psi_L^*(z) \triangleq \sum_{k=2}^{\infty} \psi_{Lk}^*\cdot z^k \quad , \quad \psi_L^{*(1)}(z) = \sum_{k=2}^{\infty} k\cdot\psi_{Lk}^*\cdot z^{k-1} \quad , \qquad (B.25)$$

where ψ_{Lk}^* are the components of the row vector $\vec{\psi_L^*}$. Equations for $S^*(z)$ and $S^{*(1)}(z)$ are obtained by substituting (B.23) together with (3.28) in (3.26). The same procedure that yielded (5.21) leads to

$$S^{*(1)}(\lambda) = \frac{Q\cdot S^*(\lambda)\cdot u^{*(1)}(\lambda) + w^{*(1)}(\lambda) + \psi_L^{*(1)}(\lambda)}{1 - v^{*(1)}(\lambda)} \quad . \qquad (B.26)$$

Hence

$$S^*(z) = 1 + Q\cdot S^*(\lambda)\cdot d^*(z) + d_w^*(z) + d_{\psi_L}^*(z) \quad , \qquad (B.27)$$

and

$$S^*(\lambda) = \frac{1 + d_w^*(\lambda) + d_{\psi_L}^*(\lambda)}{1 - Q\cdot d^*(\lambda)} \quad , \qquad (B.28)$$

where $d_{\psi_L}^*(z) \triangleq \psi_L^*(z) + K_L(\lambda)\cdot\psi_L^{*(1)}(\lambda)\cdot v^*(z)$, (B.29)

and $d^*(z)$, $d_w^*(z)$ and $K_L(\lambda)$ are defined as in (5.23), (5.24) and (5.25), respectively. □

<u>Direct method: S_N for arbitrarily biased coins</u> (cf. (5.28)).

$$S_N = 1 + Q\cdot S^*(\lambda)\cdot\sum_{k=2}^{N}\frac{N!}{(N-k)!}\cdot d_k^* + \sum_{k=2}^{N}\frac{N!}{(N-k)!}\cdot(d_{wk}^* + d_{\psi_L k}^*) \quad , \quad N \geqslant 2 \quad , \qquad (B.30)$$

with $d_{\psi_L k}^* \triangleq \psi_{Lk}^* + K_L(\lambda)\cdot\psi_L^{*(1)}(\lambda)\cdot v_k^*$, $k \geqslant 2$, (B.31)

initial values (B.5) and d_k^* and d_{wk}^* as defined in (5.29) and (5.30). □

Appendix C

Collection of numerical results

Numerical results for all four RAS's considered are given for Q=2...10 for $\overline{\alpha_Q}$, $\max_N |\alpha_Q(N)-\overline{\alpha_Q}|$, $\overline{\beta_Q}$ and $\max_N |\beta_Q(N)-\overline{\beta_Q}|$; for Q=2 and Q=3 for L_N, S_N, Var_N, E[Y], E[Y^2] and Var[Y]. The results for Q=2 mainly serve as reference.

Basic blocked access RAS's, fair coins		
Q	$\overline{\alpha_Q}$	$\max_N \|\alpha_Q(N)-\overline{\alpha_Q}\|$
2	2.885390	.00000313
3	2.730718	.0007074
4	2.885390	.005372
5	3.106675	.016608
6	3.348664	.035044
7	3.597288	.060387
8	3.847187	.092061
9	4.096077	.129451
10	4.342945	.171984

Table C.1

Modified blocked access RAS's, fair coins		
Q	$\overline{\alpha_Q}$	$\max_N \|\alpha_Q(N)-\overline{\alpha_Q}\|$
2	2.664043	.00000235
3	2.673351	.0006506
4	2.860692	.005254
5	3.093325	.016466
6	3.340442	.034899
7	3.591775	.060248
8	3.843263	.091931
9	4.093157	.129330
10	4.340697	.171872

Table C.2

Basic blocked access RAS's , fair coins , Q=2			
N	L_N	S_N	Var_N
2	5.000000	33.0000	8.0000
3	7.666667	68.5556	9.7778
4	10.523810	124.2834	13.5329
5	13.419048	197.0083	16.9375
6	16.313057	286.4268	20.3110
7	19.200922	392.3620	23.6866
10	27.853197	809.6322	33.8316
15	42.281247	1838.457	50.753
20	56.707829	3283.446	67.668

Table C.3

Basic blocked access RAS's , fair coins , Q=3			
N	L_N	S_N	Var_N
2	5.500000	37.0000	6.7500
3	7.750000	70.1875	10.1250
4	10.346154	119.9837	12.9408
5	13.080769	187.1949	16.0884
6	15.854291	270.8053	19.4467
7	18.624544	369.6995	22.8259
10	26.838887	752.9989	32.6730
15	40.431597	1683.545	48.831
20	54.103256	2992.316	65.153

Table C.4

Modified blocked access RAS's, fair coins , Q=2			
N	L_N	S_N	Var_N
2	4.500000	25.0000	4.7500
3	7.000000	54.8333	5.8333
4	9.642857	101.0374	8.0527
5	12.314286	161.7214	10.0797
6	14.984793	236.6331	12.0890
7	17.650691	325.6458	14.0989
10	25.639897	677.5413	20.1369
15	38.960935	1548.163	30.209
20	52.280872	2773.566	40.276

Table C.5

Modified blocked access RAS's, fair coins , Q=3			
N	L_N	S_N	Var_N
2	5.333333	33.8889	5.4444
3	7.583333	66.1389	8.6319
4	10.128205	113.5819	11.0013
5	12.799359	177.4194	13.5958
6	15.509441	256.9375	16.3947
7	18.219106	351.1768	19.2409
10	26.263712	717.3754	27.5928
15	39.574342	1607.374	41.246
20	52.954866	2859.215	54.997

Table C.6

Basic free access RAS's, fair coins @ λ = .3		
Q	$\overline{\beta_Q}$	$\max_N \lvert\beta_Q(N)-\overline{\beta_Q}\rvert$
2	17.248724	.0000824
3	10.736279	.0013551
4	11.479659	.010542
5	13.543532	.035511
6	16.619169	.084411
7	20.969191	.168822
8	27.286852	.309411
9	37.099198	.549130
10	54.241541	.994873

Table C.7

Basic free access RAS's, fair coins @ λ = .3		
Q	$\overline{\alpha_Q}$	$\max_N \lvert\alpha_Q(N)-\overline{\alpha_Q}\rvert$
2	2.793489	.0000022
3	2.543609	.0000761
4	2.583241	.000534
5	2.674970	.001388
6	2.776455	.002366
7	2.876133	.003198
8	2.970464	.003704
9	3.058527	.003784
10	3.140348	.003393

Table C.8

Basic free access RAS's , fair coins , Q=2			
λ	$L^*(\lambda)\equiv E[Y]$	$S^*(\lambda)\equiv E[Y^2]$	Var[Y]
.01	1.0002046	1.00169	.00128
.05	1.0056488	1.05433	.04300
.1	1.0262215	1.31981	.26668
.15	1.0711281	2.18779	1.04048
.2	1.1616706	5.16493	3.81545
.25	1.3580076	18.5442	16.7000
.3	1.9205465	137.992	134.303
.35	8.2289164	34383.4	34315.7

Table C.9

Basic free access RAS's , fair coins , Q=3			
λ	$L^*(\lambda)\equiv E[Y]$	$S^*(\lambda)\equiv E[Y^2]$	Var[Y]
.01	1.0002290	1.00189	.00143
.05	1.0061948	1.05821	.04578
.1	1.0278809	1.32048	.26394
.15	1.0726500	2.08075	.93017
.2	1.1558691	4.25722	2.92119
.25	1.3134596	11.4737	9.7485
.3	1.6524178	45.0546	42.3241
.35	2.6950807	406.044	398.781

Table C.10

Basic free access RAS's , fair coins , Q=2 @ λ = .3			
N	L_N	S_N	Var_N
2	24.204578	3392.48	2806.62
3	40.491594	6356.07	4716.50
4	57.583324	10055.44	6739.60
5	74.856766	14382.12	8778.58
6	92.139184	19308.21	10818.59
7	109.401259	24825.25	12856.62
10	161.137110	44930.60	18965.43
15	247.382311	90346.55	29148.54
20	333.625650	150638.	39331.

Table C.11

Basic free access RAS's , fair coins , Q=3 @ λ = .3			
N	L_N	S_N	Var_N
2	17.660914	1096.82	786.92
3	27.134279	1980.87	1244.60
4	37.472336	3133.70	1729.52
5	48.173471	4550.11	2229.42
6	58.991556	6215.15	2735.15
7	69.817738	8116.25	3241.73
10	102.103213	15179.31	4754.24
15	155.659932	31494.53	7264.51
20	209.365436	53613.8	9779.9

Table C.12

Modified free access RAS's , fair coins , Q=2			
λ	$L^*(\lambda) \equiv E[Y]$	$S^*(\lambda) \equiv E[Y^2]$	Var[Y]
.01	1.0001787	1.00126	.00091
.05	1.0048978	1.03976	.02994
.1	1.0224334	1.22546	.18009
.15	1.0596257	1.78979	.66698
.2	1.1310868	3.51216	2.23280
.25	1.2728475	9.80719	8.18705
.3	1.6037331	44.9064	42.3345
.35	2.8814136	690.596	682.294

Table C.13

Modified free access RAS's , fair coins , Q=3			
λ	$L^*(\lambda) \equiv E[Y]$	$S^*(\lambda) \equiv E[Y^2]$	Var[Y]
.01	1.0002206	1.00172	.00128
.05	1.0059652	1.05318	.04121
.1	1.0268245	1.29207	.23770
.15	1.0697562	1.97909	.83471
.2	1.1490826	3.91707	2.59668
.25	1.2976157	10.1760	8.4922
.3	1.6098284	37.8286	35.2370
.35	2.5123697	295.467	289.155

Table C.14

Modified free access RAS's , fair coins , Q=2 @ λ = .3			
N	L_N	S_N	Var_N
2	16.156412	1074.18	813.15
3	27.433960	2148.77	1396.15
4	39.175391	3544.21	2009.50
5	51.017180	5228.99	2626.24
6	62.862861	7194.93	3243.19
7	74.696904	9439.30	3859.67
10	110.171355	17845.64	5707.92
15	169.30633	37453.35	8788.72
20	228.4401	64054.4	11869.5

Table C.15

Modified free access RAS'S , fair coins , Q=3 @ λ = .3			
N	L_N	S_N	Var_N
2	16.554918	915.90	641.83
3	25.590214	1679.92	1025.06
4	35.359898	2676.57	1426.24
5	45.451826	3904.51	1838.64
6	55.653855	5353.21	2255.86
7	65.867838	7012.60	2674.03
10	96.345135	13205.92	3923.53
15	146.90344	27577.97	5997.35
20	197.5906	47116.8	8074.8

Table C.16

Appendix D

Regroupings for numerical computations

In this Appendix we list (without explicit derivation) some of the formulas which we derived for the purpose of numerical stability.

Blocked access RAS's

L_N in equation (3.40). We observe that for $N \geq 3$ L_N can be expressed in terms of L_{N-1} as follows

$$L_N = L_{N-1} + [Q\cdot(N-1) - \delta_M\cdot(\frac{N}{Q}-1)]\cdot\sum_{m=1}^{\infty}\cdot Q^{-m}\cdot(1-Q^{-m})^{N-2} +$$
$$+ \delta_M\cdot(1-Q^{-1})\cdot\sum_{m=0}^{\infty}\{[1-Q^{-m}\cdot(1-Q^{-1})]^{N-1} - (1-Q^{-m})^{N-2}\} , \qquad \text{(D.1)}$$

with $$L_2 = 1 + \frac{Q^2}{Q-1} - \frac{\delta_M}{Q\cdot(Q-1)} . \qquad \text{(D.2)} \square$$

Basic free access RAS's

$d^*(\lambda)$ in equations (4.22) to (4.24). From (4.23) one gets by expanding the exponential functions into power series, summing over m and substituting (4.20) and (4.12)

$$d^*(\lambda) = \frac{\lambda\cdot Q}{Q\cdot(1-\lambda)-1}\cdot e^{-\lambda\cdot\frac{Q}{Q-1}}\cdot\sum_{i=1}^{\infty}\frac{i}{i+1}\cdot[i\cdot\frac{1-Q^{-1}}{1-Q^{-i}} - Q^{-1}]\cdot(\frac{Q}{Q-1})^{i}\cdot\frac{\lambda^{i}}{i!} . \qquad \text{(D.3)} \square$$

L_N in equation (4.25). Again, we observe that for $N \geq 3$ L_N can be expressed in terms of L_{N-1} , i.e.,

$$L_N = L_{N-1} + Q \cdot L^*(\lambda) \cdot K_L(\lambda) \cdot e^{-\mu} \cdot \{ \sum_{m=0}^{\infty} \mu \cdot Q^{-m} \cdot e^{\mu \cdot Q^{-m}} \cdot [1-(1-Q^{-m})^{N-1}] + $$

$$+ (N-1) \cdot \sum_{m=1}^{\infty} Q^{-m} \cdot e^{\mu \cdot Q^{-m}} \cdot (1-Q^{-m})^{N-2} \} \quad , \quad (D.4)$$

with $$L_2 = 1 + Q \cdot L^*(\lambda) \cdot K_L(\lambda) \cdot e^{-\mu} \cdot \sum_{m=0}^{\infty} Q^{-m} \cdot e^{\mu \cdot Q^{-m}} \cdot (1+\mu \cdot Q^{-m}) \quad . \quad (D.5) \square$$

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par

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